Dynamical Teichmüller Ergodicity of Fixed Genus

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Abstract

We give relations: where biholomorphic, ergodic invariant measure $\nu_{\mathbb{T}^m}$ is generated by tori Teichmuller space of $\mathbb{R}$ uniform forest asymptotics in wild line-breaking random root-growth re-graft process, implying an everywhere modular form transformation T^N.

Keyword: Dynamical-tori, biholomorphic-map, Teichmüller-ergodicity

Section 1

1.1. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$; $k \in \mathbb{N}$, $u > 0$. Then prove unique existence

$$\int_{\overline{\mathbb{R}}} e^{-x^2/u}\, dx, \quad \int_{-a}^{b} e^{-x^2/u}\, dx, \quad (2^k u)^{\left(\frac{1}{2}\right)} \int_0^\infty t^{\left(\frac{k-1}{2}\right)} e^{-t}\, dt, \quad \int_{\overline{\mathbb{R}}} x^k\, e^{-x^2/u}\, dx.$$

1.2. Let H_N be group of Hermitian matrices. Then H_N is Euclidean space of scalar product $\langle A, B\rangle = \mathrm{Tr}(AB)$, and then $\dim(H_N)$ is well-defined.

1.3. Let dH_N be Lebesgue measure of Euclidean space by prior problem. Then the unique existence

$$\int_{H_N} e^{-\mathrm{Tr}(X^2)}\, dH_N.$$

1.4. (Gaussian unitary ensemble). Let $\left\{\xi_{jk}, \eta_{jk}\right\}_{j,k=1}^{N}$ be set of independent identically distributed (iid) Gaussian variables of zero mean, unit variance. We define a random Hermitian matrix $\left(H_{jk}\right)$ of order N as follows:

$$H_{jk} = \begin{cases} \xi_{jj} & \text{if} \quad j = k \\ \frac{1}{\sqrt{2}}\left(\xi_{jk} + i\eta_{jk}\right) & \text{if} \quad j < k \\ \frac{1}{\sqrt{2}}\left(\xi_{jk} - i\eta_{jk}\right) & \text{if} \quad j > k \end{cases}$$

then density exists for $\left(H_{jk}\right)_{j,k=1}^{N}$ with respect to dH_N.

1.5. Let (ξ_1, ξ_2, ξ_3) be uniformly distributed on surface of 2-dimensional unit sphere in 3-dimensional real (Eulidean) space. There exists distribution for the case of random variable ξ_1.

1.6. Let random variable $\xi^{(N)} = \left(\xi_1^{(N)}, \ldots, \xi_{N+1}^{(N)}\right)$ be uniformly distributed on surface of N-sphere of radius $\sqrt{N}$. There exists limiting distribution of the random variable $\xi_1^{(N)}$ when $N \longrightarrow \infty$.

1.7. There exists volume of N-dimensional sphere (in real space).

1.8. There exists surface area of $(N-1)$-dimensional sphere.

1.9. For sets $\{\phi_j\}$ and $\{\psi_j\}$ of functions, there exists proof for

$$\det\left(\left(\phi_{j-1}(x_k)\right)_{j,k=1}^N\right) \det\left(\left(\psi_{j-1}(x_k)\right)_{j,k=1}^N\right) \;=\; \det\left(\left(\sum_{\ell=1}^{N} \phi_{\ell-1}(x_j)\, \psi_{\ell-1}(x_k)\right)_{j,k=1}^{N}\right).$$

1.10. Let $S_N(x) = \frac{\sin(Nx/2)}{\sin(x/2)}$, $\forall\, x$; let symbol i denotes imaginary unit, then

$$\prod_{1\leqslant j,k\leqslant N}^{N} \left|\exp(ix_k) - \exp(ix_j)\right|^2 \;=\; \det\left(\left(S_N\left(x_k - x_j\right)\right)_{j,k=1}^{N}\right).$$

Section 2

2.1. Let kth Catalan number be:

$$C_k = \frac{1}{k+1}\binom{2k}{k} = \frac{(2k)!}{(k+1)!\,k!}.$$

Then it follows that

$$\int_{-2}^{2} x^{2k}\,\frac{1}{2\pi}\sqrt{4-x^2}\;dx \;=\; C_k.$$

2.2. Let Bernoulli random walk be integer sequence

$$\{S_k\}_{0\leqslant k\leqslant N}$$

If $S_0 = 0$ and $|S_{t+1} - S_t| = 1$, $\forall\, t \leqslant (N-1)$, then for all even $N \geqslant 2$, the number of non-negative (i.e. $S_t \geqslant 0$, $\forall\, t \leqslant N$) Bernoulli walks ending in 0 (i.e. $S_N = 0$) is Catalan number $C_{N/2}$. Hint: Use reflection principle.

2.3. Catalan numbers give generating function

$$1 + \sum_{k=1}^{\infty} C_k z^k \;=\; \frac{1-\sqrt{1-4z}}{2z}.$$

2.4. Any probability measure on the real line is obtainable as a weak limit of discrete measures (i.e. finite linear combination of delta measures).

2.5. There exists Wigner theorem for Gaussian unitary ensemble. Note: Recall proof of Wigner theorem for real symmetric Wigner matrices.

2.6. Consider Gaussian unitary ensemble of 2×2 matrices. There exists unique joint local density for eigenvalues of this random matrix.

2.7. Consider random $2{\times}2$ unitary matrix distributed as Haar. There exists unique joint density for eigenvalues of this random matrix.

Section 3

Let mes denote Lebesgue measure in the space $\mathbb{R}^d$, resp. torus $\mathbb{T}^d{=}\mathbb{R}^d/\mathbb{Z}^d$.

3.1. There exists the map

$$F\colon \mathbb{T}^2 \longrightarrow \mathbb{T}^2 \mid F(x,y) \;=\; (x{+}\alpha,\, y{+}x) \;\; (\mathrm{mod}\ \mathbb{Z}^2)$$

where α is irrational and ergodic by Lebesgue measure on torus.

3.2. Let transformation F have invariant measure ν. Then it implies:

a. There exists measure ν_b for $\nu_b(A) = \nu(h_b^{-1}(A))$, where $h_b(A) = (x,\, y+b)$ is invariant with respect to F.

b. For measurable set $E \subset \mathbb{T}^1$, $(\mathrm{mes}\, E) > 0$, the measure ν_E for

$$\nu_E(A) \;=\; \frac{1}{\mathrm{mes}\, E} \int_E \nu_b(A)\, db$$

is well-defined, and an invariant probability measure for F.

3.3. There exists that $\nu_{\mathbb{T}}$ coincides with the Lebesgue measure.

3.4. There exists that the transformation F is strictly ergodic.

3.5. The sequence $x_k{=}\{a_2k^2 + a_1k + a_0 \,|\, a_2 \notin \mathbb{Q}\}$, where curly brackets $\{\cdot\}$ denotes fraction, is uniformly distributed on unit interval;

$$\text{i.e.,}\quad \lim_{N\longrightarrow\infty} \frac{\#\big\{k\in\{0,\dots,N{-}1\},\ \forall\, x_k \in I\big\}}{N} \;=\; \mathrm{mes}\, I, \qquad \forall\, I \subset [0,1].$$

3.6. There exists the map

$$F\colon \mathbb{T}^m \longrightarrow \mathbb{T}^m \mid F(x_1,\dots,x_m) \;=\; (x{+}\alpha,\, x_2{+}x_1,\dots,x_m{+}x_{m-1}) \;\; (\mathrm{mod}\ \mathbb{Z}^m)$$

where α is irrational and ergodic by Lebesgue measure on torus.

3.7. There exists that F is strictly ergodic.

3.8. The sequence $x_k = \{a_m k^m + \cdots + a_1 k + a_0 \mid a_m \notin \mathbb{Q}\}$, where curly brackets $\{\cdot\}$ denotes fraction, is uniformly distributed on unit interval.

3.9. The sequence $x_k = \{a_m k^m + \cdots + a_1 k + a_0 \mid a_m \notin \mathbb{Q}\}$, where curly brackets $\{\cdot\}$ denotes fraction, is uniformly distributed on unit interval if at least one of the coefficients $a_1, \ldots, a_m$ is irrational.

3.10. There exists direct proof (without resorting to argument relating to mean uniform convergence) that the cyclic rotation map

$$R_\alpha \colon \mathbb{T} \longrightarrow \mathbb{T} \;\mid\; R_\alpha(x) \;=\; x + \alpha \pmod 1$$

is strictly ergodic if $\alpha \notin \mathbb{Q}$. Hint: Let interval I have length less than $1/N$. Then there exists N-pairwise non-intersecting images of $R_\alpha^{k_1}(I), \ldots, R_\alpha^{k_N}(I)$. Evaluate $\nu(I)$ for arbitrary R_α invariant measure ν.

Section 4

We denote symmetric group $\mathcal{S}_N$ i.e. set $\{1, \ldots, N\}$ group of permutations. Let segment-shift T fix segment-length vectors $(\lambda_1, \ldots, \lambda_N)$ for shift vectors $(w_1, \ldots, w_k)$, i.e. mapping $T \colon x \longmapsto x + w_k$ for lengths (λ_k) of intervals (I_k).

4.1. There exists proof that $\sum_k \lambda_k w_k = 0$.

4.2. Construct Rosy graphs for shift of 2, 3, and 4 segments.

4.3. If Keane condition (i.e. $T_{\pi,a}$-orbits of points $x_1 = a_1$, $x_2 = a_1 + a_2$, $\ldots$, $x_{N-1} = \sum_{i=1}^{N-1} a_i$, $\pi \in \mathcal{S}_N$, are pairwise disjoint and infinite) holds, the segment length decreases indefinitely in sequential application of Rosy product.

4.4. Consider 2-segment shift. Let $x = \lambda_1/\lambda_2$ be length ratio (left to right).

a. Find x transformation by Rosy induction R; solve x continued fraction.

b. For each x, there exists P such that

$$P(x) \;=\; R^{k(x)}(x)$$

is the degree of Rosy induction to go from one set to the another, where the set of shifts for the two segments is divided into two sets $A = \{x < 1\}$ and $B = \{x > 1\}$, and we are allowed to choose coordinate $y = x$ on set A, and coordinate $y = 1/x$ on set B.

4.5. Let $T\colon x \longmapsto \{1/x\}$ be Gauss-Kuzmin map where $\{\cdot\}$ denotes fraction in interval $(0,1)$. Then the measure μ with density

$$p_\mu(x) \;=\; \frac{1}{\ln(2(1+x))}$$

is an invariant measure.

4.6. Consider the Gauss-Kuzmin transformation T. Let $I_{a_1\cdots a_N}$ be segment with ends $[0,a_1,\ldots,a_N]$ and $[0,a_1,\ldots,a_{N-1},a_N+1]$. Then the map T^N from $I_{a_1\cdots a_N}$ is bijective to the whole segment $[0,1]$.

In addition, the map $\left(T^N\big|_{I_{a_1\cdots a_N}}\right)^{-1}$ is defined by

$$x \longmapsto \frac{p_N + p_{N-1}x}{q_N + q_{N-1}x}$$

where $p_k/q_k = [0,a_1,\ldots,a_k]$.

4.7. Consider the map $F = \left(T^N\big|_{I_{a_1\cdots a_N}}\right)^{-1}$.

a. Evaluate how F distorts mes, i.e. how $\mathrm{mes}(S)=\mathrm{mes}(S)/\mathrm{mes}((0,1))$ and $\mathrm{mes}(F(S))/\mathrm{mes}(I_{a_1,\ldots,a_n})$, for $S\subset(0,1)$, differ.

b. Derive ergodicity of measure μ for the Gauss-Kuzmin transformation. Hint: Choose invariant set of positive measure μ (or mes), and use Lebesgue density (point) theorem.