Dynamical Teichmüller Ergodicity of Fixed Genus

Matthew Bernard[‡]

 ‡ mattb@berkeley.edu

[‡]Algebra Computing Bio Colab (ACBC), Berkeley, CA, USA

Abstract

We characterize biholomorphic, ergodic invariant measure $\nu_{\mathbb{T}^m}$ of tori Teichmuller space, \mathbb{R} uniform forest asymptotics in wild line-breaking random root-growth re-graft process, for an everywhere modular form transformation T^n .

Keyword: Dynamical-tori, biholomorphic-map, Teichmüller-ergodicity

Section 1

1.1. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}; k \in \mathbb{N}, u > 0$. Then prove unique existence

$$\int_{\overline{\mathbb{R}}} e^{-x^2/u} \, dx, \quad \int_{-a}^{b} e^{-x^2/u} \, dx, \quad (2^k u)^{\left(\frac{1}{2}\right)} \int_{0}^{\infty} t^{\left(\frac{k-1}{2}\right)} e^{-t} \, dt, \quad \int_{\overline{\mathbb{R}}} x^k \, e^{-x^2/u} \, dx$$

1.2. Let H_N be group of Hermitian matrices. Then H_N is Euclidean space of scalar product $\langle A, B \rangle = \text{Tr}(AB)$, and then dim (H_N) is well-defined.

1.3. Let dH_N be Lebesgue measure of Euclidean space by prior problem. Then the unique existence

$$\int_{H_N} e^{-\mathrm{Tr}(X^2)} \, dH_N.$$

1.4. (Gaussian unitary ensemble). Let $\{\xi_{jk}, \eta_{jk}\}_{j,k=1}^{N}$ be set of independent identically distributed (iid) Gaussian variables of zero mean, unit variance. We define a random Hermitian matrix (H_{jk}) of order N as follows:

$$H_{jk} = \begin{cases} \xi_{jj} & \text{if } j = k \\ \frac{1}{\sqrt{2}} \left(\xi_{jk} + i\eta_{jk} \right) & \text{if } j < k \\ \frac{1}{\sqrt{2}} \left(\xi_{jk} - i\eta_{jk} \right) & \text{if } j > k \end{cases}$$

then density exists for $(H_{jk})_{j,k=1}^N$ with respect to dH_N .

1.5. Let (ξ_1, ξ_2, ξ_3) be uniformly distributed on surface of 2-dimensional unit sphere in 3-dimensional real (Eulidean) space. There exists distribution for the case of random variable ξ_1 .

1.6. Let random variable $\xi^{(N)} = \left(\xi_1^{(N)}, \ldots, \xi_{N+1}^{(N)}\right)$ be uniformly distributed on surface of *N*-sphere of radius \sqrt{N} . There exists limiting distribution of the random variable $\xi_1^{(N)}$ when $N \longrightarrow \infty$.

1.7. There exists volume of *N*-dimensional sphere (in real space).

1.8. There exists surface area of (N-1)-dimensional sphere.

1.9. For sets $\{\phi_j\}$ and $\{\psi_j\}$ of functions, there exists proof for

$$\det\left((\phi_{j-1}(x_k))_{j,k=1}^N\right)\det\left((\psi_{j-1}(x_k))_{j,k=1}^N\right) = \det\left(\left(\sum_{\ell=1}^N \phi_{\ell-1}(x_j)\,\psi_{\ell-1}(x_k)\right)_{j,k=1}^N\right).$$

1.10. Let $S_N(x) = \frac{\sin(Nx/2)}{\sin(x/2)}, \forall x$; let symbol *i* denotes imaginary unit, then $\prod_{1 \leq j,k \leq N}^{N} \left| \exp(ix_k) - \exp(ix_j) \right|^2 = \det\left(\left(S_N(x_k - x_j) \right)_{j,k=1}^{N} \right).$

Section 2

2.1. Let kth Catalan number be:

$$C_{k} = \frac{1}{k+1} \binom{2k}{k} = \frac{(2k)!}{(k+1)! \, k!}$$

Then it follows that

$$\int_{-2}^{2} x^{2k} \frac{1}{2\pi} \sqrt{4 - x^2} \, dx = C_k.$$

2.2. Let Bernoulli random walk be integer sequence

$$\{S_k\}_{0\leqslant k\leqslant N}$$

If $S_0 = 0$ and $|S_{t+1} - S_t| = 1$, $\forall t \leq (N-1)$, then for all even $N \geq 2$, the number of non-negative (i.e. $S_t \geq 0$, $\forall t \leq N$) Bernoulli walks ending in 0 (i.e. $S_N = 0$) is Catalan number $C_{N/2}$. Hint: Use reflection principle.

2.3. Catalan numbers give generating function

$$1 + \sum_{k=1}^{\infty} C_k z^k = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

2.4. Any probability measure on the real line is obtainable as a weak limit

of discrete measures (i.e. finite linear combination of delta measures).

2.5. There exists Wigner theorem for Gaussian unitary ensemble. Note: Recall proof of Wigner theorem for real symmetric Wigner matrices.

2.6. Consider Gaussian unitary ensemble of 2×2 matrices. There exists unique joint local density for eigenvalues of this random matrix.

2.7. Consider random 2×2 unitary matrix distributed as Haar. There exists unique joint density for eigenvalues of this random matrix.

Section 3

Let mes denote Lebesgue measure in the space \mathbb{R}^d , resp. torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$.

3.1. There exists the map

$$F: \mathbb{T}^2 \longrightarrow \mathbb{T}^2 \mid F(x, y) = (x + \alpha, y + x) \pmod{\mathbb{Z}^2}$$

where α is irrational and ergodic by Lebesgue measure on torus.

3.2. Let transformation F have invariant measure ν . Then it implies:

a. There exists measure ν_b for $\nu_b(A) = \nu(h_b^{-1}(A))$, where $h_b(A) = (x, y + b)$ is invariant with respect to F.

b. For measurable set $E \subset \mathbb{T}^1$, (mes E) > 0, the measure ν_E for

$$\nu_E(A) = \frac{1}{\operatorname{mes} E} \int_E \nu_b(A) \, db$$

is well-defined, and an invariant probability measure for F.

3.3. There exists that $\nu_{\mathbb{T}}$ coincides with the Lebesgue measure.

3.4. There exists that the transformation F is strictly ergodic.

3.5. The sequence $x_k = \{a_2k^2 + a_1k + a_0 | a_2 \notin \mathbb{Q}\}$, where curly brackets $\{\cdot\}$ denotes fraction, is uniformly distributed on unit interval;

i.e.,
$$\lim_{N \to \infty} \frac{\#\{k \in \{0, \dots, N-1\}, \ \forall x_k \in I\}}{N} = \operatorname{mes} I, \quad \forall I \subset [0, 1].$$

3.6. There exists the map

$$F: \mathbb{T}^m \longrightarrow \mathbb{T}^m \mid F(x_1, \dots, x_m) = (x + \alpha, x_2 + x_1, \dots, x_m + x_{m-1}) \pmod{\mathbb{Z}^m}$$

where α is irrational and ergodic by Lebesgue measure on torus.

3.7. There exists that F is strictly ergodic.

3.8. The sequence $x_k = \{a_m k^m + \cdots + a_1 k + a_0 \mid a_m \notin \mathbb{Q}\}$, where curly brackets $\{\cdot\}$ denotes fraction, is uniformly distributed on unit interval.

3.9. The sequence $x_k = \{a_m k^m + \cdots + a_1 k + a_0 \mid a_m \notin \mathbb{Q}\}$, where curly brackets $\{\cdot\}$ denotes fraction, is uniformly distributed on unit interval if at least one of the coefficients a_1, \ldots, a_m is irrational.

3.10. There exists direct proof (without resorting to argument relating to mean uniform convergence) that the cyclic rotation map

$$R_{\alpha} \colon \mathbb{T} \longrightarrow \mathbb{T} \mid R_{\alpha}(x) = x + \alpha \pmod{1}$$

is strictly ergodic if $\alpha \notin \mathbb{Q}$. Hint: Let interval I have length less than 1/N. Then there exists N-pairwise non-intersecting images of $R_{\alpha}^{k_1}(I), \ldots, R_{\alpha}^{k_N}(I)$. Evaluate $\nu(I)$ for arbitrary R_{α} invariant measure ν .

Section 4

We denote symmetric group S_N i.e. set $\{1, \ldots, N\}$ group of permutations. Let segment-shift T fix segment-length vectors $(\lambda_1, \ldots, \lambda_N)$ for shift vectors (w_1, \ldots, w_k) , i.e. mapping $T: x \longmapsto x + w_k$ for lengths (λ_k) of intervals (I_k) .

4.1. There exists proof that $\sum_k \lambda_k w_k = 0$.

4.2. Construct Rosy graphs for shift of 2, 3, and 4 segments.

4.3. If Keane condition (i.e. $T_{\pi,a}$ -orbits of points $x_1 = a_1, x_2 = a_1 + a_2, \ldots, x_{N-1} = \sum_{i=1}^{N-1} a_i, \pi \in S_N$, are pairwise disjoint and infinite) holds, the segment length decreases indefinitely in sequential application of Rosy product.

4.4. Consider 2-segment shift. Let $x = \lambda_1/\lambda_2$ be length ratio (left to right). **a.** Find x transformation by Rosy induction R; solve x continued fraction.

b. For each x, there exists P such that

$$P(x) = R^{k(x)}(x)$$

is the degree of Rosy induction to go from one set to the another, where the set of shifts for the two segments is divided into two sets $A = \{x < 1\}$ and $B = \{x > 1\}$, and we are allowed to choose coordinate y = x on set A, and coordinate y = 1/x on set B.

4.5. Let $T: x \mapsto \{1/x\}$ be Gauss-Kuzmin map where $\{\cdot\}$ denotes fraction in interval (0, 1). Then the measure μ with density

$$p_{\mu}(x) = \frac{1}{\ln(2(1+x))}$$

is an invariant measure.

4.6. Consider the Gauss-Kuzmin transformation T. Let $I_{a_1\cdots a_N}$ be segment with ends $[0, a_1, \ldots, a_N]$ and $[0, a_1, \ldots, a_{N-1}, a_N + 1]$. Then the map T^N from $I_{a_1\cdots a_N}$ is bijective to the whole segment [0, 1].

In addition, the map
$$(T^N|_{I_{a_1\cdots a_N}})^{-1}$$
 is defined by
 $x \longmapsto \frac{p_N + p_{N-1}x}{q_N + q_{N-1}x}$

where $p_k/q_k = [0, a_1, ..., a_k].$

4.7. Consider the map $F = (T^N |_{I_{a_1 \cdots a_N}})^{-1}$.

a. Evaluate how F distorts mes, i.e. how mes(S) = mes(S)/mes((0, 1)) and $mes(F(S))/mes(I_{a_1,\dots,a_n})$, for $S \subset (0, 1)$, differ.

b. Derive ergodicity of measure μ for the Gauss-Kuzmin transformation. Hint: Choose invariant set of positive measure μ (or mes), and use Lebesgue density (point) theorem.