Crumbling, Piecing, and Cracking: Naive Analysis of Global Warming

Decoupling Global Warming := More Attention to Stuff in Bra–Ket Invariant of Mathematical Physics

CRUMBLING AND PIECING

Launch into global warming[1] with Legendre, Adrien-Marie. Becoming an instant fame much early-on in the 18^{th} century for his measure of the terrestrial meridian that a meter became almost surely known as $1/10^7$ pole-equator distance, his Mathematicians-type chef-d'oeuvre: *Roots of Polynomials* — which directly inspired the achievement of Galois, and subsequent magnum opuses on elliptic integrals and quadratic reciprocity, respectively galvanizing Abel and Gauss into their achievements, and conjoined, or even necessarily connected spherical harmonics, the theory of angular momentum, without hesitation — is certainly germane to contemporary modeling of climate problems. Undeniably, angular momentum gave us blackbody[2] theory via Max Planck in 20^{th} century, and oh yes, sure Planck earned the Nobel Prize for Quantum Mechanics in 1918, an accomplishment which is far superior in many ways than Einstein's relativity, since it opens up a whole new era of measure-entropic symmetries and predictability.

Adrien-Marie Legendre (1752-1833)
`http://bit.ly/legendre`

Going by the genre at the end of the century that energy E of *the* experimentally observed (radiation[3]) is definitely quantized, i.e., no longer continuous on $[0, \infty)$, but exists only in $\gamma_{\geq 0}$-multiple of discrete minimum (quantum or photon), if $\mathbf{K}$ is a σ-algebra of a filtration of field $\mathcal{K}$, and $\gamma \leq \widetilde{\gamma} \in \gamma_{\geq 0}$-subfield $\Gamma \subseteq \mathcal{K} \mid \omega \in$ field $\mathcal{K}$ (or ω field extension $\mathcal{K} \supseteq \Gamma$), then $G\Big(f\left(\gamma, \widetilde{\gamma}\right); \omega\Big) \leftarrow (\gamma, \widetilde{\gamma}; \omega)$ discrete energy generator, for partition function $f(\gamma, \widetilde{\gamma}) \mid \Gamma \times \Gamma \to \Gamma$, is a $\mathbf{K}_\Gamma \times \mathbf{K}$, where $\mathbf{K}$ is measurable map from $\Gamma \times \mathcal{K}$ into $\mathcal{K}$ in the distribution sense lawed by the $\gamma_{\geq 0}$-relation:

$$\begin{aligned} E_\gamma &\propto G\Big(f\left(\gamma, \widetilde{\gamma}\right); \omega\Big) \\ \text{i.e.,}\quad E_\gamma &= h\, G\Big(f\left(\gamma, \widetilde{\gamma}\right); \omega\Big) \end{aligned}$$

for proportionality constant $h \in$ field $\mathcal{K}$.

In particular,

$$\begin{aligned} \widetilde{\gamma} &= 1 + 1 + \cdots (\gamma \text{ times}) \text{ in } \mathcal{C}\text{ompact } \mathcal{M}\text{anifold} \\ \widetilde{\gamma} &= \mathbb{1}_{i \geq 0}\Big(\lfloor \cdot \rfloor \cup \lceil \cdot \rceil\Big) + \mathbb{1}_{i \geq 0}\Big(\lfloor \cdot \rfloor \cup \lceil \cdot \rceil\Big) + \cdots \\ &\quad \cdots + (\gamma \text{ times}) \text{ in } \mathcal{N}\text{on-}\mathcal{C}\text{ompact } \mathcal{M}\text{anifold}. \end{aligned}$$

On the (not necessarily compact) closed interval $[\omega, \omega + d\omega]$,

$$dE = d\gamma \cdot \mathbb{E}[E] \tag{1}$$

Thus, for infinitely smooth algebra $\mathbb{C}^\infty(\mathbf{K}) \mid \langle \mathcal{K}, \mathbf{K} \rangle \leq \langle \mathcal{K} \mid \Omega \rangle$, where $\Omega :=$ Borel σ-algebra of field $\mathcal{K}$, the set $\big\{z \in \mathbf{Z}\left(\mathbb{C}^\infty(\mathbf{K})\right)\big\}$ is almost surely $\mathcal{O}$pen $\mathcal{M}$anifold bundle from the homeomorphism

$$\langle \cdot, \cdot \rangle : \mathbf{Z}^{-1}(\mathbf{K}) \times \mathbf{Z}^{\star -1}(\mathbf{K}) := \Big\{\gamma : \mathbf{Z}^{-1}(\gamma) \in \mathbf{K},\ \mathbf{Z}^{\star -1}(\gamma) \in \mathbf{K}\Big\}.$$

For discrete E_γ of γ-colored indices on bialgebra $\widetilde{\gamma}$ assortments, where $\big|\mathbf{Z}\left(\mathbb{C}^\infty(\mathbf{K})\right)\big| = 2^{\widetilde{\gamma}}$, $\big|\mathbf{Z}\left(\mathbb{C}^\infty(\mathbf{K})\right)\big| \leq 2^{|\mathbf{K}_\Omega|}$ with the latter being power-set cardinality of the $\langle \mathcal{K}, \Omega \rangle$ discrete subspace

$$\mathbf{K}_\Gamma \bigcap \left\{ \underbrace{\Big\lfloor \Big\langle z \,\Big|\, z^\star \Big\rangle \Big\rfloor}_{\xi \text{ likely copies}} \bigoplus \underbrace{\Big\lceil \Big\langle z \,\Big|\, z^\star \Big\rangle \Big\rceil}_{\eta \text{ likely copies}} \right\}.$$

That is, combinatorial look casts its furtive scope as usual, and it is almost unbearably embarrassing just to glance around $\mathbf{Z}\left(\mathbb{C}^\infty(\mathbf{K})\right)$, even as in obligatory denial of a heraldry real villain knight $\mathcal{K}$ of castle Ω, where Γ longs for discrete works to fix castle lock σ. Therefore, going by set of discrete E_γ partition (not surely stationary) works, an embodiment of open bundle-fiber in underlying $\mathcal{K}$ is lawed by

$$\lim_{\gamma \to \infty} \left(\sup_{\gamma \in \Gamma} \bigcap_{\gamma} \Big\{ \mathbb{E}\Big[E_\gamma^{(\cdot)} \mid \mathbf{K}_\Gamma\Big] \Big\} \right) \leq \sup_{\Omega/\mathcal{K}} \left(\inf_{\gamma \in \Gamma} \mathbb{E}\Big[\Big\{E_\gamma^{(\cdot)} \mid \mathbf{K}_\Omega\Big\}\Big] \right).$$

Thus, by one-one correspondence with infinitely many disjoint sequences $E_{\widetilde{\gamma}} := \Big\{E_\gamma^{(\widetilde{\gamma})}\Big\}$, all partition function is complete in Ω.

Since by the mean value[4] theorem

$$\mathbb{E}[E] := \left(\sum_\gamma \exp\left\{-\frac{E_\gamma}{K_B T}\right\} \right)^{-1} \sum_\gamma E_\gamma \exp\left\{-\frac{E_\gamma}{K_B T}\right\}$$

[1] increment in global mean temperature.

[2] a hypothetically perfect body in which all incidented e-m radiation is completely absorbed, and in all wavelength spectra and all directions, maximum possible emission is realized.

[3] infinite number of harmonically oscillating dipoles.

[4] in continuous term, this will be the equivalent of

$$\mathbb{E}[E] := \left(\int_\omega \int_\lambda \exp\left\{-\tfrac{E}{K_B T}\right\} d\omega d\lambda \right)^{-1} \int_\omega \int_\lambda E \exp\left\{-\tfrac{E}{K_B T}\right\} d\omega d\lambda$$

where $\exp\left\{\frac{h\omega}{k_BT}\right\} := 1+\left(\frac{h\omega}{k_BT}\right)+ \cdots$ higher powers > 1

$$\mathbb{E}[E] := \frac{h\omega}{\exp\left\{\frac{h\omega}{k_BT}\right\}-1} \approx \frac{h\omega}{\left(1+\frac{h\omega}{k_BT}\right)-1} := k_BT\,.$$

That is, suppose one-octant-sphere rotation of two potential states of standing waves in cubic box, as observed by *Rayleigh*:

$$d\gamma = \frac{2(4\pi)V\omega^2 d\omega}{c^3}, \quad \text{with cubic volume } L^3 = V, \quad L \in \mathbb{R}^+$$

energy density $\rho(\omega) = \frac{dE}{V} = d\gamma \cdot \mathbb{E}[E] \cdot \frac{1}{V}$

such that discrete energy is given by:

$$dE = d\gamma \cdot \mathbb{E}[E] = \rho(\omega)\cdot V = \frac{h\omega}{\exp\{h\omega/(k_BT)\}-1}\cdot V$$

where γ is an integer; ω is frequency; λ is wavelength; $c := 2.998\times10^8$, speed of light in vacuum; $h := 6.626\times10^{-34}$, Planck's fundamental constant of Quantum Mechanics.

Thus, blackbody radiation energy density is born of these forms:

1). As a function of ω and T (Planck's Law):

$$E(\omega,T) = \beta_1\omega^3\frac{1}{\exp\{\beta_2\omega/T\}-1}$$

where $\beta_1 = \frac{2h}{c^2}$, $\beta_2 = \frac{h}{k_B}$, and k_B = Boltzmann constant $1.3806503 \times 10^{-34}$ (the fundamental constant in statistical mechanics).

2). As a function of λ and T:

$$E(\lambda,T) = \frac{\alpha_1}{\lambda^5}\frac{1}{e^{\frac{\alpha_2}{\lambda T}}-1} \sim \begin{cases} \frac{\alpha_1}{\lambda^5}e^{-\frac{\alpha_2}{\lambda T}} \to 0, \text{ for } \lambda\to 0 \\ \frac{\alpha_1}{hc\lambda^4}K_BT \to 0, \text{ for } \lambda\to\infty \end{cases}$$

where $\alpha_1 = 2\pi hc^2$, $\alpha_2 = \frac{hc}{k_B}$.

Integrating $E(\lambda,T)$ along all paths $[0,\frac{1}{T})$ over all the wavelengths (and all angles of the hemisphere):

$$\lim_{\tau\to0}\int_0^{\frac{1}{\tau}}\frac{\alpha_1}{\lambda^5}\frac{1}{e^{\frac{\alpha_2}{\lambda T}}-1}d\lambda \equiv \frac{\alpha_1}{\alpha_2^4}T^4\int_0^\infty x^3\frac{1}{e^x-1}dx\text{ , with } x=\frac{\alpha_2}{\lambda T}$$

$$= \frac{\alpha_1}{\alpha_2^4}T^4\sum_{k=1}^\infty\int_0^\infty x^3e^{-kx}dx\text{ , since } \frac{1}{e^x-1} = \frac{e^{-x}}{1-e^{-x}} = \sum_{k=1}^\infty e^{-kx}$$

$$= \cdots(\text{integration by part})\cdots = \frac{\alpha_1}{\alpha_2^4}T^4\sum_{k=1}^\infty\frac{6}{k^4} = \frac{6\alpha_1}{\alpha_2^4}T^4\sum_{k=1}^\infty\frac{1}{k^4}$$

$$= \frac{6\alpha_1}{\alpha_2^4}T^4\zeta(4) = \frac{\alpha_1}{\alpha_2^4}T^4\frac{\pi}{15} = \frac{2\pi hc^2}{\left(\frac{hc}{k_B}\right)^4}T^4\frac{\pi^4}{15} = \frac{2\pi^5k_B^4}{15h^3c^2}T^4$$

where ζ is the Riemann zeta function[5], thus:

$$\int_0^\infty E(\lambda,T)d\lambda = \sigma T^4\,.$$

That is, Stefan-Boltzmann Law; simply put, density of energy emitted is directly proportional to the 4^{th} power of temperature; $\sigma :=$ Stefan-Boltzmann constant 5.67×10^{-8}.

Optimizing $E(\lambda,T)$, by its $\partial_\lambda(\cdot)=0$:

$$\partial_\lambda E(\lambda,T) = \frac{\partial}{\partial\lambda}\left(\frac{\alpha_1}{\lambda^5}\frac{1}{e^{\frac{\alpha_2}{\lambda T}}-1}\right) = 0 \;\Rightarrow\; \frac{d}{du}\left(\frac{u^5}{e^u-1}\right) = 0,$$

$$\text{where } u := \frac{\alpha_2}{\lambda T} \quad\Rightarrow\quad f(u) = 1-e^{-u}-\frac{u}{5} = 0\,,$$

we have a transcendental equation $f(u)$ of known range $(-\infty,0)$. Its numerical solution by Newton-Raphson iterative process[6] (as $f'\neq0$, and f'' not changing sign on finite $[a,b]$) gives an approximate solution 4.965114. Thus,

$$\lambda_{max} = \frac{\alpha_2}{uT} = \frac{hc}{4.965114k_BT} = 2898\mu m\left(\frac{1}{T}\right).$$

That is, Wien's Law: Maximum wavelength is inversely proportional to the surface temperature.

Hence, assuming solar heating has negligible effect on earth's interior, a blackbody theory-based sun's temperature T_s and earth's surface temperature T_{be} can be estimated from an equilibrium[7] and topological standpoint:

Assuming classical equilibrium between sun's emitted radiation and its luminosity L, in space:

$$\sigma T_s^4 4\pi R^2 \approx L \approx \psi 4\pi D^2$$

where R = sun's equatorial radius 6.955×10^8, ψ = solar constant measured from space to be 1370, and D = distance between earth and sun 1.496×10^{11}, and $\approx$ denotes that 4 is as usual for a perfect sphere of which the sun is not[8] -- see NASA JPL (Updated 2010, January 6).

Thus:

$$T_s = \left(\frac{\psi D^2}{\sigma R^2}\right)^{\frac14} \approx 5782\text{ K} = 5509\ {}^o\text{C}\,.$$

Assuming, again, equilibrium between the earth's emitted and absorbed radiations, at the earth's optical "membrane", i.e., the atmosphere, which loses or gains heat by radiation only (since outer space is vacuum!) and acts as fundamental boundary condition for the flow of energy (photon) from the sun to earth:

$$\sigma T_{be}^4(4.0034\pi r^2) = \psi(1-\alpha)(\pi r^2)$$

[5] it may be helpful to know that whenever m is even, $\zeta(m)\propto\pi^m$.
[6] other methods exit in which $\{f_n\}$ converges faster.
[7] Kirchhoff's law – at equilibrium, emissivity equals absorptivity.
[8] as computed by Hugh Hudson of UC Berkeley and his team to a groundbreaking precision of 0.001% level.

where T_{be} = blackbody-earth temperature; r = earth's radius; α = earth surface albedo, i.e., proportion of incidented radiation (from sun) that is reflected back to space (mostly by cloud but also by surface, aerosols, and gases), about 0.3 or 30% today but 0.7 or 70% in ice/snow.

$4.0043\pi r^2$ (not $4\pi r^2$) accounts for the fact that earth's emission occurs over the entire sphere and the earth is not a perfect sphere (Loeb et al, 2009).

$(1 - \alpha)$ accounts for the proportion of incidented radiation that is not reflected back into outer space but serves to augment incoming radiation from the sun.

πr^2 accounts for the fact that the earth absorbs radiation in the form of a circle, essentially the shadow that would be cast by earth, which comes in from just one side.

Thus:

$$T_{be} = \left(\frac{\psi(1-\alpha)}{4.0034\sigma}\right)^{\frac{1}{4}} \approx 255 \text{ K} = -18\ ^o\text{C}.$$

N.B.: This is $\ll$ 288 K = 15 ^{o}C, the mean observed temperature T_{oe}; quite unlike mars (Ohring et al, 1962) which has relatively small atmosphere and blackbody temperature seems to be in very good agreement with observed temperature.

Thus, begging the question: Could there be some amplifying factor in the system (earth)?

Find out what is going on by first exploring the spectra: Figure 2.

Where as the normalized plot in Fig. 2 is obtained by dividing each spectrum by its maximum value so they both peak at 1, Fig. 1 is over $\lambda = 0.1 - 5\ \mu$m for sun and over $\lambda = 0.1 - 100\ \mu$m for earth, since Wien's law ⇒ Sun's $\lambda_{max} \approx 0.5\ \mu$m (from T_s = 5782 K = 5509 ^{o}C) and $\lambda_{max} \approx 11\ \mu$m for earth (from T_{be} = 255 K = −18 ^{o}C), and in standard perspective, sun's emission is about 10 to the 6th power of earth's emission.

Of particular interest is the spectral independence that can be deduced from these plots; i.e., spectrally, the sun and the earth are almost completely separated – just only about 1% overlap between them, a great advantage because solar emission and terrestrial emission can hence be treated as separate from each other to a very good approximation.

Equivalently, from these plots (and from Wien's law), solar emission is predominantly in the UV and visible regions of the

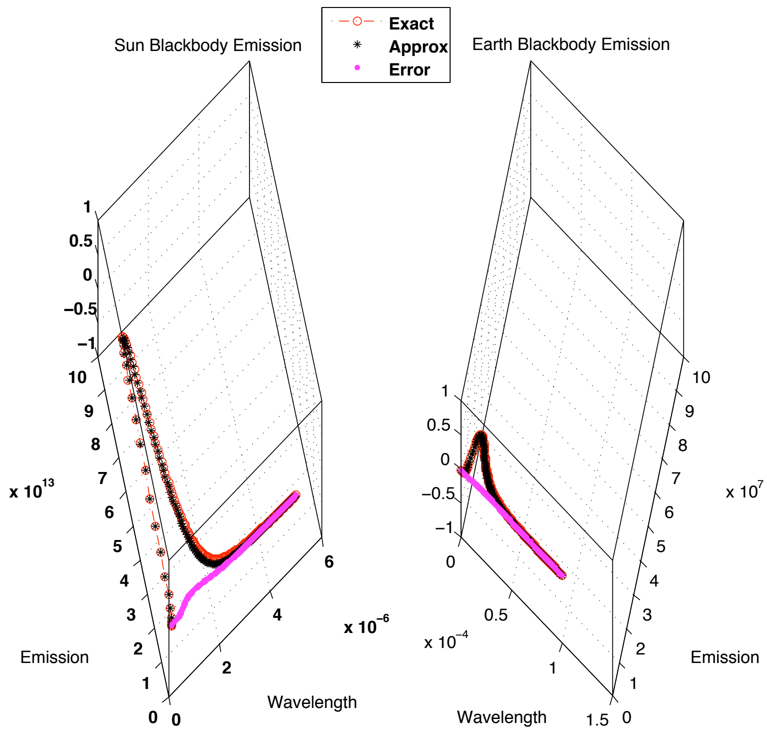


Figure 1: *C*lassical blackbody emission spectra of sun and earth.

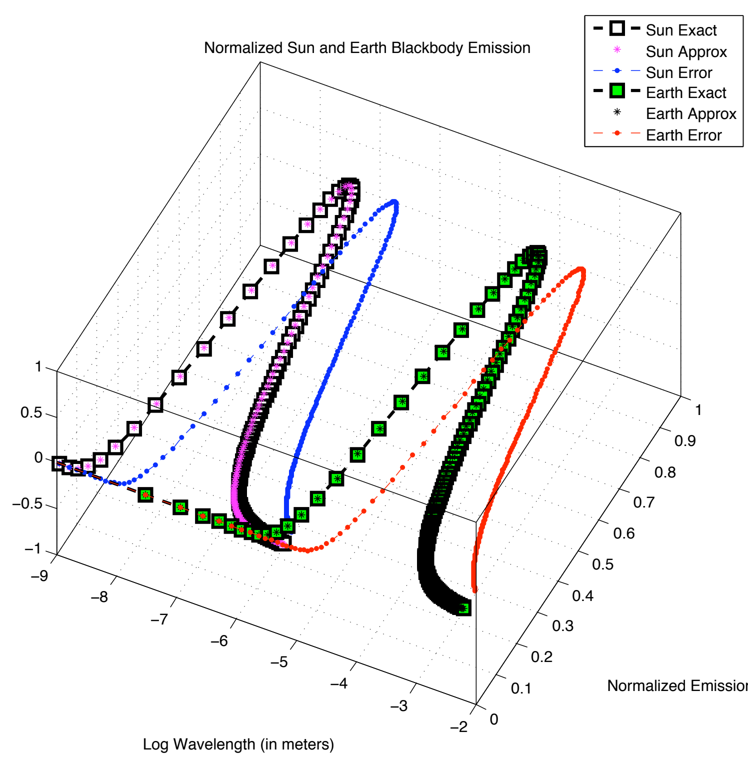
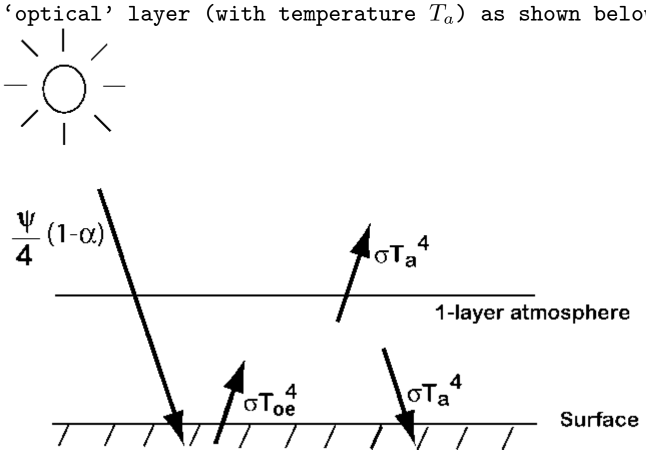


Figure 2: *N*ormalized blackbody emission of sun and earth.

spectrum ($\approx 10^{-9} - 10^{-6}$ m) while terrestrial emission is predominantly in the infrared ($\approx 10^{-6} - 10^{-3}$ m), which perhaps support the presence of a greenhouse and ensuing effects[9] in the atmosphere.

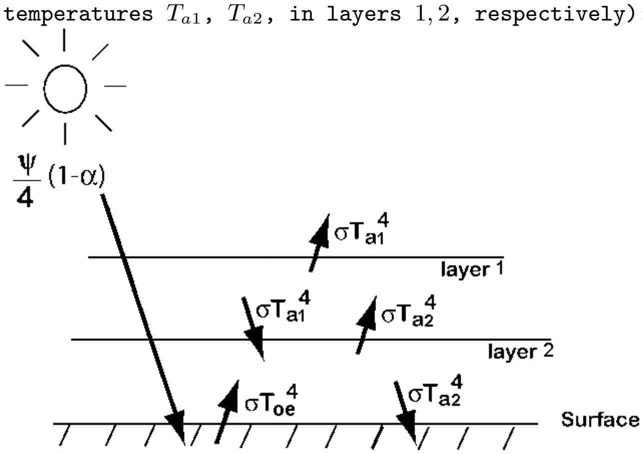
Now, supposing atmospheric greenhouse effect actually amplifies earth's surface temperature, then, try capture the factor Λ of amplication (by the so-called greenhouse effect) by relating it to the number of "optical" layers into which the atmosphere may be subdivided as follows:

First case scenario, assume atmosphere is a single 'optical' layer (with temperature T_a) as shown below:



$$\left.\begin{array}{l} \frac{\psi(1-\alpha)}{4.0034} = \sigma T_a^4 \quad \text{at top of atmosphere} \\ \text{which} \Rightarrow T_a = T_{be} \text{ from equation ()} \\ \sigma T_{oe}^4 = \frac{\psi(1-\alpha)}{4.0034} + \sigma T_a^4 \quad \text{at earth surface} \end{array}\right\} \Rightarrow T_{oe} \stackrel{?}{=} 2^{\frac{1}{4}} T_{be}$$

Second case scenario, assume a two-layer atmosphere (with temperatures T_{a1}, T_{a2}, in layers $1, 2$, respectively):



$$\left.\begin{array}{l} \frac{\psi(1-\alpha)}{4.0034} = \sigma T_{a1}^4 \quad \text{at layer 1} \\ \frac{\psi(1-\alpha)}{4.0034} + \sigma T_{a1}^4 = \sigma T_{a2}^4 \quad \text{at layer 2} \\ \sigma T_{oe}^4 = \frac{\psi(1-\alpha)}{4.0034} + \sigma T_{a2}^4 \quad \text{at earth surface} \end{array}\right\} \Rightarrow T_{oe} \stackrel{?}{=} 3^{\frac{1}{4}} T_{be}$$

[9] i.e., that the atmosphere is composed of gases like CO_2, water vapor, ozone, methane, etc., which are largely "transparent" to solar radiation (i.e., infrared radiation) but not to terrestrial radiation (i.e., visible and UV), and therefore traps energy to amplify the earth surface temperature.

$$\vdots \qquad \vdots \qquad \vdots$$

nth case scenario, assume $n > 2$ layers (with temperatures $T_{a1}, T_{a2}, \cdots, T_{an}$ in layers $1, 2, \cdots, n$, respectively):

$$\left.\begin{array}{ccc} \vdots & \vdots & \vdots \end{array}\right\} \Rightarrow T_{oe} \stackrel{?}{=} (n+1)^{\frac{1}{4}} T_{be}$$

So that the '?' mark above the '=' sign implies: Is there any $n \geq 1$ such that

$$\Lambda = (n+1)^{\frac{1}{4}} = \frac{T_{oe}}{T_{be}} \quad ?$$

But unfortunately there is no such n; not even the least possible senario $n = 1$, the one-layer atmosphere, where $T_{oe} = 2^{\frac{1}{4}} \times 255$ K ≈ 303 K, still > 288 K.

Thus, begging the other (contradictory) question: Could there be some dampening factor in the system (earth)?

CRACKING

Inspired by the seemingly small number $2^{\frac{1}{4}} \approx 1.1892$, which literally sums up all widespread debates about the effect of changing greenhouse gases and all uncertainties about the poor approximation of the Stefan-Boltzmann law, let emitted radiation E_E from the earth be based on a parameterization (rather than the bad ol' Stefan-Boltzmann law):

$$E_E = A + BT$$

such that A and B := empirical constants derived from satellite data as in Farrell (1990), taken here as 203.3 and 2.09, respectively, and $T := T(\phi)$ is the local temperature of the earth surface at a given latitude ϕ.

And, at the same ϕ, assuming this form of absorbed radiation:

$$E_S = Sf(T) = S(1 - \alpha(T))\,\xi(\phi)$$

which is already incorporating the albedo feedback as:

$$\alpha(T) = \begin{cases} 0.6 & \text{if} \quad T \leq -10^o\text{C} \quad \text{(ice-covered)} \\ 0.3 & \text{if} \quad T > -10^o\text{C} \quad \text{(ice-free)} \end{cases}$$

and ice-line ϕ_{ice} is where $T(\phi_{ice}) = T(ice) = -10$.

$$S = \frac{\psi}{4.0034} = \frac{1370}{4.0034} = 342.2,$$

$\xi(\phi)$ = insolation, i.e., amount of solar radiation reaching the given ϕ of earth surface.

then, it's either a question of $E_S = E_E$ (at equilibrium) or $E_S \neq E_E$ (outside-equilibrium).

At equilibrium (which is the initial steady state),

$$E_E = E_S \Rightarrow A + BT = Sf(T) \tag{2}$$

$$\text{i.e.,} \quad -f(T)\frac{dS}{dT} = -B + Sf'(T)$$

Outside-equilibrium, $T(\phi) := T_{equilibrium} + \delta T(\phi)$, $\delta T(\phi) > 0$, where when some perturbation (say double CO_2) is applied, outgoing energy (from earth) must be allowed to "feedback" and re-establish a new steady state.

Note the word **feedback**. So far constructed with intent of an albedo-only – whose reference block (i.e., "gold standard" against which feedback effect is to be evaluated) is imagined as the system with zero feedback – feedback also include water vapor, lapse rate, and cloud, mainly (IPCC, 2007), all of which is held fixed here (and will be discussed later on). The most important distinction to note here is that perturbation "pushes" climate from an original state to a new state, and feedback then responds (i.e., not occurring in stable climate nor occurring on its own) by either "further pushing" climate "in the direction" of perturbation (i.e., amplifying or positive feedback) or "pulling" climate "against the direction" of perturbation to bring climate closer to its original state (i.e., dampening or negative feedback). Thus, feedback essentially acts to amplify or dampen perturbation, contrasting a lot with just an ordinary perturbation-adjustment (besides simultaneously acclimatizing) response purportedly hailed in many texts.

Hence, a convergence term or energy difference $\vec{G} = -E_E + E_S$ (with which feedback and hence stability are interdependent) must be introduced as follows:

$$\vec{G} = -A - BT + Sf(T) = -\mathcal{M}_c\frac{dT}{d\phi} = -\mathcal{M}_c\ \nabla \cdot T$$

which is what Physicists call Fourier's law of heat conduction[10], where $\mathcal{M}_c$ = the meridional energy capacity per surface area of the entire earth, and $\vec{G}$ is understood in hyperbolic Brownian motion as the Poisson kernel, i.e., the density function of a probability distribution.

Thus, first-order wrt T,

$$-\mathcal{M}_c\frac{d}{d\phi}\delta T(\phi) = \left(-B + Sf'(T)\right)\delta T(\phi) \quad \Rightarrow \quad \frac{d}{d\phi}\delta T = -C\delta T$$

$$\text{where } C = \frac{f(T)}{\mathcal{M}_c}\frac{dS}{dT}$$

Moreover, $\vec{G}$ is continuously smooth (i.e., continuously differentiable) with compact support almost everywhere on the earth surface (which is simply connected), and $\vec{G}$ is conservative (i.e., path-independent)[11]. Thus, there exists some potential function g such that a convergence (i.e., density of net meridional flux) is given by

$$\nabla \cdot g = \vec{G}$$

[10] simply put, energy change or flow is directly proportional to temperature gradient.

[11] because its circulation around any closed borel subset $\partial\phi \subset \partial B_r$ of the earth surface is zero, i.e., $\int_{\partial B_r} \vec{G} \cdot ds = \int_{\partial\phi} (\nabla \times g)\, dA = \int_{\partial\phi} 0 dA = 0$.

Clearly, such g is given by:

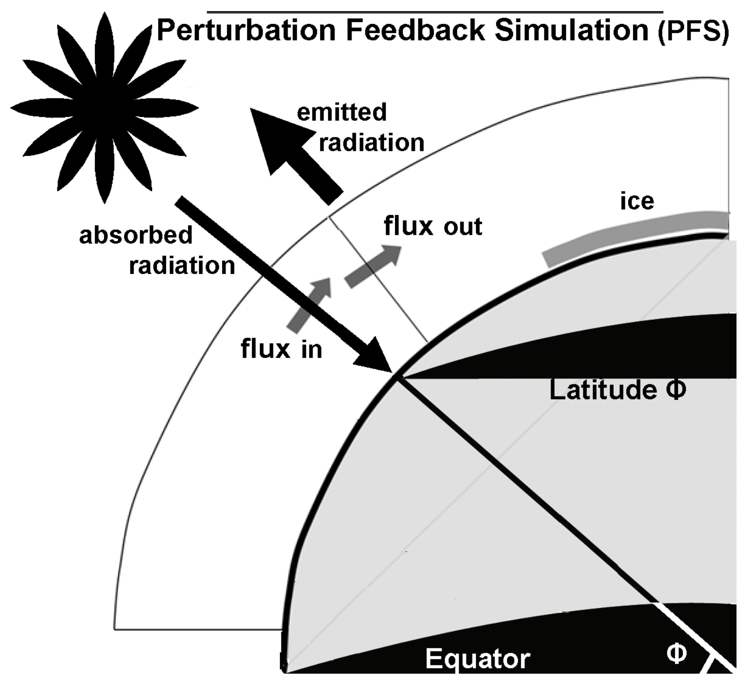
$$g(T) = AT + \frac{1}{2}BT^2 - S\int_0^T f(T')dT'$$

In fact, such $g(T)$ is the sink term, i.e., would always decrease with time, based on the chain rule:

$$\frac{dg}{d\phi} = \frac{dg}{dT}\cdot\frac{dT}{d\phi} = -\mathcal{M}_c\left(\frac{dT}{d\phi}\right)^2$$

where the last equality bears the fundamental qualitative analysis of stability, even for finitely many amplitude of perturbations, giving how large a perturbation needs be in order to push the system over the hill, out of the shallow minimum representing the present climate, into a fathomless minimum corresponding to the unfathomable freeze.

The next logical sequence is to consider a latitudinal domain $(\phi, \phi + \delta\phi)$, $\delta\phi > 0$, with nice approximation for $\nabla \cdot g$ as follows:



Let the total flux into the domain of $\delta\phi$ (i.e., across latitude ϕ over all longitudes) be given by:

$$2\pi r\cos(\phi)\left(\mathcal{M}_c\frac{dT}{rd\phi}\right)(\phi)$$

where $2\pi r\cos(\phi)$ = circumference or length of latitude belt and $rd\phi$ = width of latitude belt

Similarly, let the total flux out of the domain of $\delta\phi$ (i.e., across latitude $\phi + \delta\phi$ over all longitudes) be given by:

$$2\pi r\cos(\phi + \delta\phi)\left(\mathcal{M}_c\frac{dT}{rd\phi}\right)(\phi + \delta\phi)$$

where $2\pi r\cos(\phi+\delta\phi) =$ distance around earth at $\phi+\delta\phi$

By first-order Taylor expansion

$$\cos(\phi+\delta\phi) \ \approx\ \cos(\phi) - \delta\phi\sin(\phi)$$

and

$$\left(\mathcal{M}_c\frac{dT}{rd\phi}\right)(\phi+\delta\phi) \ \approx\ \left(\mathcal{M}_c\frac{dT}{rd\phi}\right)(\phi) + \delta\phi\frac{d}{d\phi}\left(\mathcal{M}_c\frac{dT}{rd\phi}\right)(\phi).$$

Thus,

$$\begin{aligned}
\nabla\cdot g \ &= \\
&2\pi r\cos(\phi)\left(\mathcal{M}_c\frac{dT}{rd\phi}\right)(\phi) - 2\pi r\cos(\phi+\delta\phi)\left(\mathcal{M}_c\frac{dT}{rd\phi}\right)(\phi+\delta\phi) \\
&\approx\ 2\pi r\delta\phi\sin(\phi)\left(\mathcal{M}_c\frac{dT}{rd\phi}\right)(\phi) - 2\pi r\delta\phi\cos(\phi)\frac{d}{d\phi}\left(\mathcal{M}_c\frac{dT}{rd\phi}\right)(\phi) \\
&\quad + 2\pi r(\delta\phi)^2\sin(\phi)\frac{d}{d\phi}\left(\mathcal{M}_c\frac{dT}{rd\phi}\right)(\phi) \\
&\approx\ 2\pi\delta\phi\sin(\phi)\left(\mathcal{M}_c\frac{dT}{d\phi}\right)(\phi) - 2\pi\delta\phi\cos(\phi)\frac{d}{d\phi}\left(\mathcal{M}_c\frac{dT}{d\phi}\right)(\phi)
\end{aligned}$$

where the last $\approx$ is valid in the limit that $\delta\phi\to 0$.

Dividing by area of the domain, which approaches $2\pi r^2\delta\phi\cos(\phi)$ in the limit $\delta\phi\to 0$ implies:

$$\nabla\cdot g \ \approx\ \mathcal{M}_c\frac{\tan(\phi)}{r}\frac{dT}{rd\phi} - \frac{d}{rd\phi}\left(\mathcal{M}_c\frac{dT}{rd\phi}\right). \tag{3}$$

Substituting $x=\sin(\phi) \ \Rightarrow\ dx=\cos(\phi)d\phi$ implies:

$$\mathcal{M}_c\frac{\tan(\phi)}{r}\frac{dT}{rd\phi} = \frac{\mathcal{M}_c}{r^2}\sin(\phi)\frac{dT}{\cos(\phi)d\phi} = \frac{\mathcal{M}_c}{r^2}x\frac{dT}{dx}$$

and

$$\begin{aligned}
&-\frac{d}{rd\phi}\left(\mathcal{M}_c\frac{dT}{rd\phi}\right) = -\frac{d}{d\phi}\left(\frac{\mathcal{M}_c}{r^2}\cos(\phi)\frac{dT}{\cos(\phi)d\phi}\right) \\
&= -\cos(\phi)\frac{d}{d\phi}\left(\frac{\mathcal{M}_c}{r^2}\frac{dT}{\cos(\phi)d\phi}\right) + \sin(\phi)\frac{\mathcal{M}_c}{r^2}\frac{dT}{\cos(\phi)d\phi} \\
&= -\cos^2(\phi)\frac{d}{\cos(\phi)d\phi}\left(\frac{\mathcal{M}_c}{r^2}\frac{dT}{\cos(\phi)d\phi}\right) + \sin(\phi)\frac{\mathcal{M}_c}{r^2}\frac{dT}{\cos(\phi)d\phi} \\
&= -\left(1-x^2\right)\frac{d}{dx}\left(\frac{\mathcal{M}_c}{r^2}\frac{dT}{dx}\right) + x\frac{\mathcal{M}_c}{r^2}\frac{dT}{dx}.
\end{aligned}$$

Thus, (3) $\ \Rightarrow\ -\left(1-x^2\right)\dfrac{d}{dx}\left(\dfrac{\mathcal{M}_c}{r^2}\dfrac{dT}{dx}\right) + 2x\dfrac{\mathcal{M}_c}{r^2}\dfrac{dT}{dx}$

which $\ \Rightarrow\ -\dfrac{d}{dx}\left(k\left(1-x^2\right)\dfrac{dT(x)}{dx}\right).$

Thus:

$$\nabla\cdot g \ \approx\ -\frac{d}{dx}\left(k\left(1-x^2\right)\frac{dT(x)}{dx}\right), \quad \text{for } k=\frac{\mathcal{M}_c}{r^2}.$$

Now, the nonseasonal perturbation feedback simulation (PFS) equation which takes into account the topology and surface albedo feedback of earth is given by:

$$\underbrace{A+BT(x)}_{\substack{\shortparallel \\ E_E}} + \underbrace{\left(-\frac{d}{dx}\left(k\left(1-x^2\right)\frac{dT(x)}{dx}\right)\right)}_{\substack{\shortparallel \\ \texttt{convergence}}} - \underbrace{S(1-\alpha)\xi(x)}_{\substack{\shortparallel \\ E_S}} = 0$$

with boundary condition imposed by $-k\sqrt{1-x^2}\,\dfrac{dT}{dx}=0\Big|_{x=0,\mp 1}$ (4)

where k may be thought off as a function of latitude, so as to take into account a circulation pattern known as the Hadley Cell, i.e., the more effective redistribution of energy in the tropics than in the extra-tropics (Farrell, 1990), implemented as follows:

$$k(x) = 0.45\left(1+9*\exp\left\{-\left(\frac{x}{\sin(30^o)}\right)^6\right\}\right)$$

crudely represented by increasing k by a factor of 10 (say, equatorwards of about 30^o).

In particular, thinking of k as a free parameter to be empirically adjusted, k as constant implies:

$$k\ \times\ \text{the operator } \tau := -\frac{d}{dx}(1-x^2)\frac{d}{dx}$$

understood in Quantum Mechanics as 1-particle spherical harmonic oscillator differential operator corresponding to the eigen-equation $\tau p_i(x) = i(i+1)p_i(x)$ where $p_i(x) =$ associated Legendre polynomial.

Writing

$$A+BT(x)+i(i+1)kT(x) - S(1-\alpha)\xi(x) = 0, \quad i\geq 0 \tag{5}$$

the ice-albedo feedback implies nonlinearity, which really implies the set of analytic multiple solutions as follows:

First recognizing the vector space X of T as a 2-dimensional Euclidean $\mathcal{R}^2 = \ell^2(\mathbb{N})$, the set of all square-summable sequences:

$$\left\{(a_i)_{i=1}^{\infty}\ \middle|\ \sum_{i=1}^{\infty}|a_i|^2 < \infty\right\}$$

whose inner (or scalar) product[12] is the standard dot product:

$$\left\langle t\ ,\ p\right\rangle = \lim_{n\to\infty}\sum_{i=0}^{n} t_i p_i(x)^* = \lim_{n\to\infty}\sum_{i=0}^{n} t_i p_i(x)\,.$$

[12] the continuous map $\langle\cdot\ ,\ \cdot\rangle : T\times T\to\mathcal{R}\in\mathbb{R}$.

That is, X is a special case Hilbert space[13] $\mathcal{H} = L^2([-1,1], dx)$, the set of all square-integrable functions on $[-1,1]$:

$$\left\{ (f_i(x))_{i=1}^{\infty} \,\middle|\, \int_{[-1,1]} |f_i(x)|^2 < \infty \right\}$$

whose inner (or scalar) product is given by:

$$\left\langle t\,,\,p \right\rangle = \int_{[-1,1]} t(x)p(x)^* d\mu(x) = \lim_{n\to\infty} \sum_{i=0}^{n} t_i p_i(x)^* = \lim_{n\to\infty} \sum_{i=0}^{n} t_i p_i(x)\,.$$

Switching right-away to orthogonal axes, which means, using Rodrigues' formula

$$p_i(x) = \frac{1}{2^i i!} \frac{d^i}{dx^i} \left(x^2 - 1\right)^i \quad \text{for each } i \geq 0 \tag{6}$$

implies a class of orthogonal polynomials, $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = \frac{1}{2}(3x^2 - 1)$, $p_3(x) = \frac{1}{2}(5x^3 - 3x), \cdots$ called Legendre polynomials, essentially orthogonal[14] basis that all lie in the open interval[14] $(-1,1)$ such that p_i has i distinct real roots.[14]

Now, since every polynomial of degree $n < \infty$ can be expressed as a linear combination of the Legendre polynomials, write:

$$T(x) \;=\; \lim_{n\to\infty} \sum_{i=0}^{n} t_i p_i(x) \;=\; \left\langle t\,,\,p \right\rangle \;\in\; L^2([-1,1], dx) \tag{7}$$

where all i's are restricted to even integers in order to preserve $p_i(-x) = p_i(x)$, i.e., South-North symmetry of climate distribution, and ensure that the boundary condition is guaranteed for any level of truncation.

Following then Huybers (2006) integrated summer insolation (which is based on observed data), defining $\xi(x)$ as annual mean distribution function of insolation so that pole-to-pole global area-mean is unity:

$$1 \;=\; \frac{1}{2} \left\langle\, \xi(x) \,\right\rangle \;=\; \widehat{\xi} \;\in\; L^2([-1,1], dx)$$

and approximating bell curve distribution of $\xi(x)$ using Legendre polynomials up to p_2, since quadratic gives closer fitting (than quartic) to fit the "real" bell curve distribution, implies:

$1 = \xi_0 p_0 + \xi_2 p_2$, excluding odd terms ξ_1, p_1 for symmetry sake.

Thus, $\xi_0 = 1$ and ξ_2 not so obvious; and, ξ_2 is determined by heuristically shrinking or expanding to fit; e.g., $\xi_2 = -0.482$ produces a best fit of $\approx 98\%$ accuracy as shown in Fig. 3.

[13] because a vector space of open ball $B_r(T) = \{y \in X | d(T, y) < r, r > 0\}$ has an inner (or scalar) product and is also Cauchy; i.e., "complete"; i.e., has a limit.

[14] from the orthogonal relation $\langle t\,,\,p\rangle = 0 \in L^2([-1,1], dx)$.

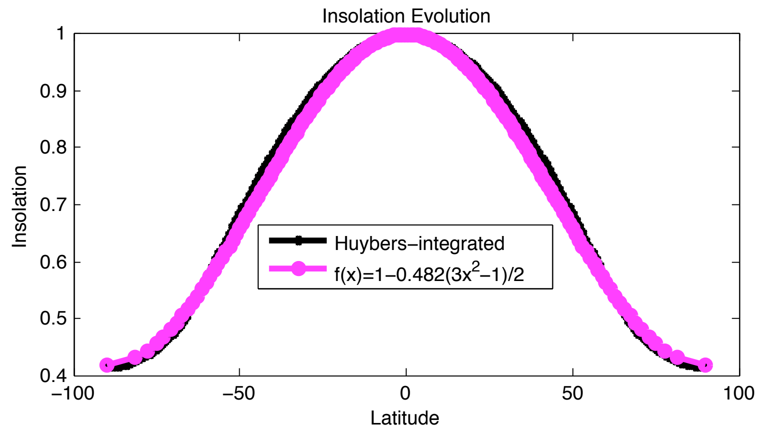


Figure 3: *N*ormalized annual mean insolations as a function of latitude, where Huybers-Integrated has been adjusted for solar constant 1370; Huybers (2006) uses 1365.

Hence,

$$\xi(x) \approx 1 - 0.482 p_2(x) = 1 - 0.482 \left(\frac{3x^2 - 1}{2}\right) = 1.241 - 0.723x^2\,.$$

Thus, write (5) as:

$$\frac{-Ap_0 + S(1-\alpha)(1.241 - 0.723x^2)}{i(i+1)k + B} = T(x) = \left\langle t\,,\,p \right\rangle, \quad \texttt{for } i \geq 0 \tag{8}$$

$$\begin{aligned} \text{so that} \quad \left\langle T(x)\,,\,p_i(x) \right\rangle &= \left\langle \left(\lim_{n\to\infty} \sum_{j=0}^{n} t_j p_j(x) \right), p_i(x) \right\rangle \\ &= \lim_{n\to\infty} \sum_{j=0}^{n} t_j \left\langle p_j(x)\,,\,p_i(x) \right\rangle \\ &= \lim_{n\to\infty} \sum_{j=0}^{n} t_j \frac{2}{2n+1} \delta_{ij} \\ &= t_i \frac{2}{2n+1} \end{aligned}$$

$$\texttt{where } \frac{2}{2i+1}\delta_{ij} \;=\; \left\langle p_i\,,\,p_j \right\rangle \;\in\; L^2([-1,1], dx)$$

such that as tensor form of the identity matrix,

$$\texttt{the Kronecker delta } \delta_{ij} := \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \quad 0 \leq i, j < \infty\,.$$

That is, all the Fourier-Legendre amplitudes $t_i \in L^2([-1,1], dx)$, are given as follows:

$$t_i \;=\; \frac{2n+1}{2} \left\langle \left(\frac{-Ap_0 + S(1-\alpha)(1.241 - 0.723x^2)}{i(i+1)k + B} \right), p_i(x) \right\rangle. \tag{9}$$

That is,

$$i=0 \quad\Rightarrow\quad t_0 = \frac{-A}{B} + \frac{S(1-\alpha)}{2B}\int_{-1}^{1}\left(1.241-0.723x^2\right)dx$$

$$i=2 \quad\Rightarrow\quad t_2 = \frac{5S(1-\alpha)}{12k+2B}\int_{-1}^{1}\left(1.241-0.723x^2\right)p_2(x)\,dx$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

`truncating at` $i=m$:

$$t_m = \frac{(2m+1)S(1-\alpha)}{2m(m+1)k+2B}\int_{-1}^{1}\left(1.241-0.723x^2\right)p_m(x)\,dx$$

`so that` (7) `becomes` $T(x) \approx t_0 + t_2p_2(x) + \cdots + t_mp_m(x)$

Now, notice $i=0$ implies $A + Bt_0 = S * 0.7$. That is, equation (2) and

$$t_0 = \frac{1}{2}\Big\langle T(x)\,,\,p_0\Big\rangle = \frac{1}{2}\Big\langle\,T(x)\,\Big\rangle = \widehat{T}$$

implies that, in baseline state (α `= 0.3,` A `= 203.3,` B `= 2.09,` S `= 342.2, and only albedo feedback and nothing else`), the earth's mean PFS (global) temperature must be $t_0 \approx 17\ ^o\mathrm{C} = 290$ K, which is pretty close to 288 K (the mean observed).

Moreover m could be extended to 4, since quartic-approximation is also parabolic. But it's better to truncate at $i=2$, because, for $i \geq 4$, the associated integral in (9) becomes zero; obvious from the orthogonal relation[15]. Thus, it can be expected that quartic-approximation will give a very small amplitude.

Besides, there are finitely many steady-state solutions of T (and $\widehat{T}$), with respect to k (which was earlier left as an adjustable parameter). Thus, k must be fixed in order to compute $t_{i\geq 2}$. The idea is truncate (7) at $i=2$. Then, using observed pole-to-equator temperature difference, i.e., observed $T(x=0) - T(x=1) = \Delta T_{p-e}$, which will be taken as 34 ^{o}C (although varies) so solution can easily converge, compute k as follows:

$$T(x) \approx t_0 + t_2p_2(x) \approx 17 + t_2\left(\frac{3x^2-1}{2}\right)$$

$$\Rightarrow t_2 \approx -\frac{2}{3}\Delta T_{p-e} \approx -\frac{2}{3}\Big(T(x=0)\ -\ T(x=1)\Big) \approx -\frac{2}{3}*34\,.$$

Thus, from ($i=2$) above:

$$k \ \approx\ -\frac{B}{6} + \frac{5S(1-\alpha)\int_{-1}^{1}\left(1.241-0.723x^2\right)\left(\frac{3x^2-1}{2}\right)dx}{-8\Delta T_{p-e}}$$

$$\Rightarrow \quad k \ \approx\ 0.5\ .$$

[15] $\int_{-1}^{1} h(x)p_i(x)dx = 0$ for every polynomial h of degree $< i$, and Legrendre polynomial p_i.

Now, for $T(x) = t_0 + t_2p_2(x) = t_o + t_2\left(3x^2-1\right)$, given t_0, t_2 and k, the nonseasonal – i.e., equation (4) – analytical scheme is done; first, by evenly discretization of the parameterized latitudinal domain x^n into N points:

$$x_1 = -1 + \frac{\Delta x}{2}, \quad x_N = 1 - \frac{\Delta x}{2}, \quad \text{and} \quad x_{(n+1)} - x_n = \Delta x = \frac{2}{N}$$

`where underscript index identifies discrete instance.`

Approximating by centered linear finite differences:

$$\left.\frac{dT}{dx}\right)_{x_n} = \frac{1}{2}\left(\left.\frac{dT}{dx}\right)_{x_{\left(n+\frac{\Delta x}{2}\right)}} + \left.\frac{dT}{dx}\right)_{x_{\left(n-\frac{\Delta x}{2}\right)}}\right)$$

$$= \frac{1}{2}\left(\frac{T_{(n+1)} - T_n}{\Delta x} + \frac{T_n - T_{(n-1)}}{\Delta x}\right) = \frac{T_{(n+1)} - T_{(n-1)}}{2\Delta x}$$

and

$$\left.\frac{d^2T}{dx^2}\right)_{x_n} = \frac{1}{\Delta x}\left(\left.\frac{dT}{dx}\right)_{x_{\left(n+\frac{\Delta x}{2}\right)}} - \left.\frac{dT}{dx}\right)_{x_{\left(n-\frac{\Delta x}{2}\right)}}\right)$$

$$= \frac{T_{(n+1)} - 2T_n + T_{(n-1)}}{\left(\Delta x\right)^2}\,.$$

Thus, re-writing the energy difference part of (4) as follows:

$$-\frac{d}{dx}\left(k\left(1-x^2\right)\frac{dT}{dx}\right) \ =\ 2xk\frac{dT}{dx} - (1-x^2)k\frac{d^2T}{dx^2}$$

gives the finite difference form:

$$2x_nk\left(\frac{T_{(n+1)} - T_{(n-1)}}{2\Delta x}\right) - \frac{\left(1-(x_n)^2\right)k\left(T_{(n+1)} - 2T_n + T_{(n-1)}\right)}{(\Delta x)^2}$$
$$= \ S(1-\alpha)\xi(x_n) - A - BT_n\,.$$

Further rearranging the terms:

$$-\left(\frac{x_nk}{\Delta x} + \frac{\left(1-(x_n)^2\right)k}{\left(\Delta x\right)^2}\right)T_{(n-1)} + \left(B + \frac{2k\left(1-(x_n)^2\right)}{\left(\Delta x\right)^2}\right)T_n$$
$$+\left(\frac{x_nk}{\Delta x} - \frac{\left(1-(x_n)^2\right)k}{\left(\Delta x\right)^2}\right)T_{(n+1)} \ =\ S(1-\alpha)\xi(x_n) - A\,.$$

Using simplified notations:

$$q_n = \frac{kx_n}{\Delta x}, \quad h_n = \frac{-\left(1-(x_n)^2\right)k}{\left(\Delta x\right)^2}$$

the system to solve is given by:

$$\left(h_n - q_n\right)T_{(n-1)} \ +\ \left[B - 2h_n\right]T_n \ +\ \left(h_n + q_n\right)T_{(n+1)}$$
$$= \ S(1-\alpha)\xi(x_n) - A$$

which can be expressed in matrix algebra as:

$$\mathbf{VT} = \mathbf{U}$$

or in tensor generalization as:

$$\sum_{m=1}^{n} V_{n,m} T_m = U_n$$

where $P_n = S(1-\alpha)\xi(x_n) - A$, and, entries $V_{n,m}$ of matrix V are given by:

$$V_{n,m} = \begin{cases} h_n - q_n & \text{if} \quad m = n-1 \\ B - 2h_n & \text{if} \quad m = n \\ h_n + q_n & \text{if} \quad m = n+1 \\ 0 & \text{if} \quad \text{otherwise} \end{cases} .$$

That is,

$$\begin{pmatrix} V_{1,1} & V_{1,2} & 0 & & & & & \vdots & 0 \\ V_{2,1} & V_{2,2} & 0 & 0 & 0 & & & \vdots & 0 \\ 0 & V_{3,2} & 0 & 0 & & & & \vdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \vdots & 0 & V_{n,n-1} & V_{n,n} & V_{n,n+1} & 0 & \vdots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \vdots & & & 0 & V_{N-2,N-3} & V_{N-2,N-2} & V_{N-2,N-1} & 0 \\ 0 & \vdots & & & & 0 & V_{N-1,N-1} & V_{N-1,N-1} & V_{N-1,N} \\ 0 & \vdots & & & & & 0 & V_{N,N-1} & V_{N,N} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ \cdots \\ T_n \\ \cdots \\ T_{(N-2)} \\ T_{(N-1)} \\ T_N \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ \cdots \\ U_n \\ \cdots \\ U_{(N-2)} \\ U_{(N-1)} \\ U_N \end{pmatrix} .$$

Now, eliminating ocean domain (by not having E_S depend on temperature), incorporate a loop of iterations $z = 1 : 1000$ such that, for each iteration, Δx is reduced by 0.5 and the series of iteration must stop at some $z \leq 1000$ once the amplitude between successive iterations is less than 10^{-12}. And, then, the converged solution at each node (latitudinal point) is spilled out for temperature, including the global mean temperature $\widehat{T} = \frac{1}{N}\sum_N T$, alongside the net meridional flux, and each of the three energy terms in (4), as shown in Fig. 4.

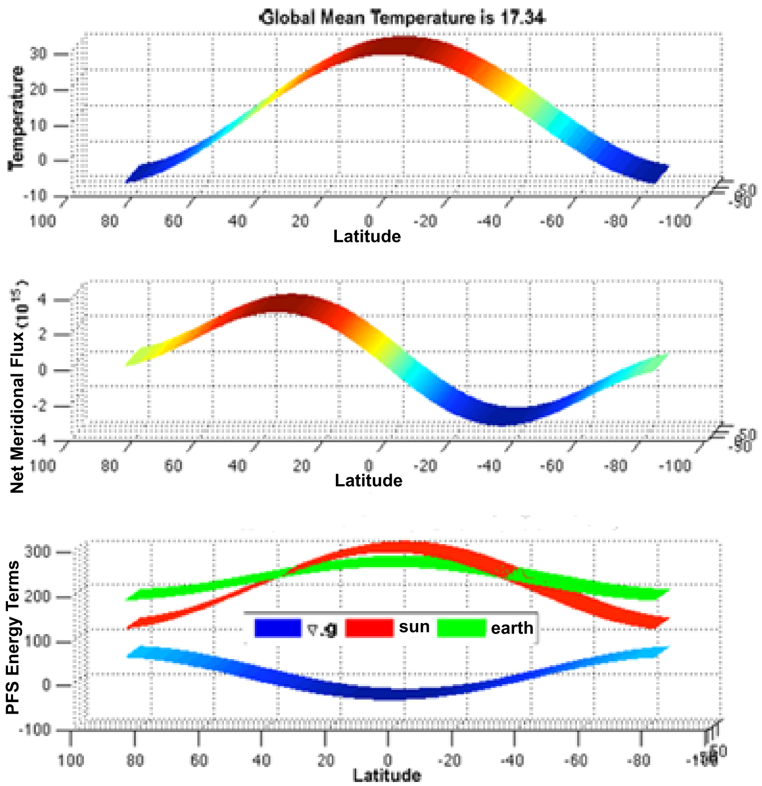


Figure 4: `Temperature, Net Meridional Flux, PFS Energy Terms, as a function of Latitude, in baseline state (`α` = 0.3, `A` = 203.3, `B` = 2.09, `S` = 342.2, and only albedo feedback and nothing else.`

DISCUSSION

• 1). Pole-to-equator temperature difference $\approx 34\ ^oC$ (consistent with earlier assumption for finding t_2 and hence k). • Net meridional flux maximum at $\approx 3.3\times10^{15}$ and goes to zero at equator $x = 0$ and at poles $x = \mp1$, satisfying boundary condition $\sqrt{1-x^2}\frac{dT}{dx} = 0$. • 2). $\nabla\cdot g$ curve does have a kink at ice line (ϕ around $T = -10\ ^oC$). • 3). Simulating the Hadley Cell (i.e., k becomes a function of the latitude) shrinks ΔT_{p-e}. • 4). Decrease (resp. increase) in k corresponds to increase (resp. decrease) in ΔT_{p-e} and increase (resp. decrease) in net meridional flux. But these changes are not simply proportional and change in k does not change $\widehat{T}$ (unknown why). • 5). When $k = 0$ (i.e., no meridional energy transport or flux), $\widehat{T}$ is ≈ 8.77, $\Delta T_{p-e} \approx 105$, convergence $\nabla \cdot g = 0$, while the sun E_S and the earth E_E are equal all times. • 6). Last but not least, the system is less sensitive to changes in S when ice line (which is at $\approx$ 72N) is fixed, i.e., albedo feedback is off.

Moreover, snowball earth has much weaker Hadley circulation and jet stream (Pierrehumbert, 2005), which corresponds to smaller k and larger Δt_{p-e} in the timeline. Thus, given evidence of extensive (maybe total) glaciations in over half a billion years ago, there is a high plausibility that glaciation may have been broken by times of relative warmth across latitudinal belts.

Now, $\widehat{T} = 15.1\ ^oC$, which is approximately the mean observed $15\ ^oC$, corresponds to a k of ≈ 0.386805 (at baseline $\frac{S}{S_o} = 1$), with a $T_{\max}$ of $\approx 30\ ^oC$ and a $T_{\min}$ of $\approx -19\ ^oC$. Hence take k as 0.386805 from now on, assuming albedo is the only feedback in the system and nothing else changes. Thus:

1). Perturbation due to solar density ΔS may be thought up in percentage and thus computed as follows:

$$\%\Delta S = \left(\frac{S}{S_o} - 1\right) \times 100\%,$$

where S_o = baseline, and S = new, and at baseline $\frac{S}{S_o} = 1$

Suppose $-10\ ^oC$ still denotes when it is cold enough for ice to form (i.e., equatorial temperature $T_{max} \leq -10\ ^oC$ is required for total glaciation, and polar temperature $T_{min} > -10\ ^oC$ is required for complete deglaciation) and linear models of solar evolution suggest that solar density has gradually increased by roughly 10% over the last 10^9 (1 billion) years. Then, varying S at the mean observed temperature of $15\ ^oC$ (i.e., k value of 0.386805) shows that:

• Roughly 11.34087742991-2 % decrease in S is required to have a total glaciation. Thus, predicting snowball earth to about 1.134×10^9 years ago (roughly around when the earth was 3.366 billion years old).

• Roughly 1.447976683797-8 % increase is required for complete deglaciation. Thus, there is still at least 144 million years to come before there will be no more ice left on the earth surface.

• Roughly 0.5% change in solar density, i.e., 50 million years is required for a $1\ ^oC$ rise in $\widehat{T}$, which is actually very stable and implies that the perturbation due to a realistic change in solar density is very small, thus, solar evolution (which is almost surely a flat-line besides the 11-year Schwabe cycle oscillatory signal) is not responsible for most of the observed "late-20th century warming" and nothing to argue for a low sensitivity here.

Mindful of the verity that numbers are even likely to change with more feedbacks in the system, these predictions are no better than what is specified. In fact, depending on how solar density perturbation is defined, its formula above can often be adjusted to account for ozone absorption of UV and other factors.

2). A painless and pleasant way of introducing CO_2 perturbation into the system is to adjust A by an amount ΔA given by:

$$\Delta A = -k * \ln\left(\frac{[CO_2]}{360}\right), \text{ for some constant } k = 5.35\ W/m^2,$$

`where `$360 =$` today's (reference) and `$[CO_2] =$` unknown concentration in parts per million volume, Myhre et al (1998),` so that increase in atmospheric CO_2 decreases outgoing radiation, and the earth must globally warm up (in order to re-establish a new steady state).

Thus:

• The logarithmic relation ⇒ fractional change in CO_2 is what is important. Adding 1 ppmv of CO_2 to a reference concentration of 2 would produce a much larger effect than adding 1 ppmv to a reference concentration of 100 ppmv, but would produce the same effect as doubling of CO_2 from todays value, a relation deemed fit over related mundane conditions. However, perturbation is stronger than logarithmic relation at very low or very high concentrations.

• Doubling CO_2 from today's value of ∼360 ppmv to around 720 ppmv (which they say is expected to happen within the next 70 years) would produce a $\Delta A = -5.35 \ln\left(\frac{720}{360}\right) = -3.7083$ which corresponds to an increase in $\widehat{T}$ of about 4.02 ^{o}C, in roughly 200-million years (using the solar perturbation timeline calculated above).

• Compared to 360 ppmv $[CO_2]$ (today) where $A = 203.3$, 200 ppmv $[CO_2]$ (in glacial period) corresponds to $A = 206.4447$ and 280 ppmv $[CO_2]$ (in interglacial period) corresponds to $A = 204.6445$. That is, a $\widehat{T}$ rise of about 3.23 from glacial period to date, and 1.48 from interglacial period to date, with a glacial-interglacial $\widehat{T}$ rise of about 1.75.

However, paleoclimate evidence (when you go back in time far enough) suggests much larger climate changes have occurred than is possible under realistic perturbation scenarios. Amplitude of glacial-interglacial rise in T_{oe}, for instance, is in the order of 4-6 ^{o}C, and thus argues for cyclicality (since the earth could have been boiling by now if an amplitude of this order has been going on).

With this plausibility of high sensitivity and cyclicality, it makes sense to now discuss the rest piece of whole feedback picture.

Let there be an "all-inclusive" system (i.e., containing all possible feedbacks) in which all feedback (e.g., water vapor, lapse rate, and cloud, $\cdots$, and any lump-sum like cloud + sea ice, $\cdots$), except albedo, is indexed in $i \in \mathcal{I} = (1, 2, \cdots)$, so that albedo is allowed to vary freely alongside temperature, and reference block is imagined as erstwhile albedo-only feedback system (in which all feedbacks but albedo are held fixed). Moreover, let there be a λ as metric, i.e., global climate sensitivity (which is the temperature response of the global system per unit perturbation). Allowing E_E and E_S to vary freely based on the parameterization and formulation above (rather than the highly erroneous Stefan-Boltzmann law), suppose the relation $E = E_S - E_E$ holds so that at equilibrium or initial steady state (as earlier stated) $E = 0$, but when perturbed by ΔE_P (small enough compared to E_E, E_S), in order to attain a new steady state (i.e., zero sum of energy), the system must change by an amount ΔE $(= -\Delta E_P = \Delta E_S - \Delta E_E)$.

Thus, for the reference system, a temperature change ΔT_0, following a ΔE_P perturbation, winds up in:

$$\Delta T_0 \left(\frac{\Delta E_E}{\Delta T_0} - \frac{\Delta E_S}{\Delta T_0} \right) = -\Delta E = \Delta E_P \quad \Rightarrow \quad \Delta T_0 = \widehat{\lambda} \Delta E_P$$

$$\text{for some } \widehat{\lambda} = \left(\frac{\Delta E_E}{\Delta T_0} - \frac{\Delta E_S}{\Delta T_0} \right)^{-1}, \ \texttt{where } \alpha \texttt{ is } T\texttt{-independent}$$

$$\texttt{but } \alpha = \alpha(T) \quad \Rightarrow \quad \Delta T_0 = \lambda \Delta E_P$$

$$\texttt{where } \lambda = \frac{1}{\left(\frac{\Delta E_E}{\Delta T_0} - \frac{\Delta E_S}{\Delta T_0} \right) + S \frac{d}{dT}(\alpha)}$$

$$= \frac{\left(\frac{\Delta E_E}{\Delta T_0} - \frac{\Delta E_S}{\Delta T_0} \right)^{-1}}{1 \ + \ S \left(\frac{\Delta E_E}{\Delta T_0} - \frac{\Delta E_S}{\Delta T_0} \right)^{-1} \frac{d}{dT}(\alpha)} = \frac{\widehat{\lambda}}{1 + S \widehat{\lambda} \frac{d}{dT}(\alpha)}$$

Now, define $j \mid_{\forall j \neq i} = j \in \mathcal{I} \mid \forall i$ as the fixing of climate fields (besides temperature, albedo, and the i^{th} climate field). Then, for "all-inclusive" system, the temperature change ΔT, following ΔE_P perturbation, is in a Taylor series extrapolation (where $\xi(x) \longrightarrow$ unity) as follows:

$$\frac{\Delta E_E}{\Delta T_0} \cong \frac{\Delta E_E}{\Delta T} + \left.\frac{\Delta E_E}{\Delta E_{E1}}\right)_{j|j\neq 1} \frac{\Delta E_{E1}}{\Delta T} + \left.\frac{\Delta E_E}{\Delta E_{E2}}\right)_{j|j\neq 2} \frac{\Delta E_{E2}}{\Delta T} + \cdots$$

$$\text{i.e., } \frac{\Delta E_E}{\Delta T_0} \Delta T \ - \ \sum_i \left.\frac{\Delta E_E}{\Delta E_{Ei}}\right)_{j|j\neq i} \frac{\Delta E_{Ei}}{\Delta T} \Delta T \ \cong \ \Delta E_E$$

$$\frac{\Delta E_S}{\Delta T_0} \cong \frac{\Delta E_S}{\Delta T} + \left.\frac{\Delta E_S}{\Delta E_{S1}}\right)_{j|j\neq 1} \frac{\Delta E_{S1}}{\Delta T} + \left.\frac{\Delta E_S}{\Delta E_{S2}}\right)_{j|j\neq 2} \frac{\Delta E_{S2}}{\Delta T} + \cdots$$

$$\text{i.e., } \frac{\Delta E_S}{\Delta T_0} \Delta T \ - \ \sum_i \left.\frac{\Delta E_S}{\Delta E_{Si}}\right)_{j|j\neq i} \frac{\Delta E_{Si}}{\Delta T} \Delta T \ \cong \ \Delta E_S$$

$$\text{so that} \quad \Delta T - \sum_i f_i \Delta T \cong \lambda \Delta E_P$$

$$\text{i.e., } \Delta T = \frac{\lambda E_P}{1 - \sum_i f_i} = \frac{1}{1-f} \Delta T_0$$

$$\texttt{for some } f = \sum_i f_i \ \texttt{ where } f_i = i^{th} \texttt{ feedback given by:}$$

$$f_i \equiv \lambda \times \left\{ \left.\frac{\Delta E_E}{\Delta E_{Ei}}\right)_{j|j\neq i} \frac{\Delta E_{Ei}}{\Delta T} - \left.\frac{\Delta E_S}{\Delta E_{Si}}\right)_{j|j\neq i} \frac{\Delta E_{Si}}{\Delta T} \right\}$$

Thus:

• $\lambda = \left(\frac{1}{1-f} \left[\frac{\Delta E_P}{\Delta T} \right] \right)^{-1}$ ⇒ the global climate sensitivity λ is a function of (i.e., depends on) all possible feedbacks that can occur (including water vapor, lapse rate, cloud, and albedo). Thus, λ is very different from the definition implied in some bogus demonstration of low climate sensitivity (and simultaneous claim that the net effect of climate feedbacks is to dampen the so-called Planck's response) in which λ is equated to some $(-\lambda_{planck})^{-1}$, where λ_{planck} is the Planck feedback parameter (computed as the derivative of the above-forbidden Stefan-Boltzmann law):

$$\lambda_{planck} = \left(\frac{d}{dT}\left(\sigma T^4\right)\right) = 4\sigma T^3$$

• Moreover, $\Delta T = \lambda \Delta E_P \Rightarrow$ that a high sensitivity amplitude means that it is easy to change the global mean temperature, while a low sensitivity amplitude would require an enormous perturbation to get that same change. Yet, it would be hard to bring an ocean to a boil quickly, and as earlier stated, if CO_2 is instantly doubled or instantly stopped, a full temperature response will not just show up right away. It will take time. Meaning, this is an equilibrium formula deeply entwined in all possible feedbacks that can occur.

Accordingly, speculation of ΔT due to a perturbation would depend very much on understanding the distinct feedback and the total process as a whole. IPCC (2007) has extensive information on the climate feedbacks, but just to get started off:

1). Water Vapor: Broadly defined as the "amount of water that air can hold", the important thing to note is that it is the most powerful positive feedback (making the climate much more sensitive to perturbation), besides that it absorbs visible radiation, mostly resides in high altitudes where it is dry and cold, and increases nearly exponentially with temperature following Clausius-Clapeyron equation (Pierrehumbert, 2005):

$$P(T) = P(T_0) * \exp\left\{-x * \left(\frac{1}{T} - \frac{1}{T_0}\right)\right\}$$

`where` $P(T)$ `and` $P(T_0)$ `are a saturation vapor pressures at temperatures` T `and` T_0 `respectively), and` $x =$ `latent heat of phase transition divided by gas constant.`

2). Lapse Rate: Essentially the rate of decrease with height for water vapor, it tends to offset water vapor feedback, resulting in warmer temperatures at high altitudes, more water vapor, i.e., more condensation, and lifting to higher altitudes so that any given layer of the atmosphere radiates more efficiently.

3). Surface Albedo: Introduced early in the simulation, this feedback is based on change in surface (and hence reflectivity) that accompanies climate change. The best example is with sea ice. Sea ice tends to increase (or decrease) in a cooling (or warming) earth and are more reflective than surrounding ocean or land, resulting in positive feedback. Thus, this feedback is one large component of "polar amplification" which explains why climate change is more sensitive to high latitudes than to lower latitudes. Warm (or cold) climates are hence characterized by weak (strong) pole-to-equator temperature difference. Less ice in a global warming situation means more solar absorption, which ends up resulting in higher surface air temperatures. Same idea applies to a once forested area that has now become a desert.

4). Clouds: Perhaps the largest source of uncertainty in quantifying the extent of climate feedbacks that the scope of this paper may be too small to explore the various hypotheses and evidence in much better detail, Cloud influence depends heavily on latitude, optical thickness, and a host of other issues. Its very robust influence is felt in the parity competition between reflecting sunlight (low clouds mostly) and influencing the outgoing infrared radiation (high clouds mostly). It is still not clear how the parity weighs out, and thus the magnitude and even the sign of the feedback is not well constrained.

Suffice to say climate feedbacks operate dependently on one another (still independent of the output ΔT), and each takes a fraction of ΔT_0 (reference response), and feeds it back into the system (making perturbation a function of ΔT_0). E.g., response due to increase in CO_2 (which decreases E_E) tends to increase temperature, which tends to increase water vapor, which, in turn, produces a perturbation in the downwelling E_E that amplifies original perturbation. Stronger water vapor feedback would mean warmer temperatures, still less ice, still lower albedo, etc. The overall effect behaves in a power-series like fashion, $1 + f + f^2 + f^3 \cdots = 1/(1-f) = G_f$, with small and diminishing gains as time progresses, so that ΔT is related to the "sum of all possible feedbacks" f and the "gain factor" G_f as follows:

$$\Delta T = G_f \Delta T_0 = \frac{1}{1-f}\Delta T_0$$

`where` $\lambda \Delta E_P = \Delta T_0$ `is the reference partition response`

$0 < G_f < 1$ `for` $f < 0 \Rightarrow$ `a positive overall feedback`
$G_f > 1$ `for` $0 < f < 1 \Rightarrow$ `a negative overall feedback`
`and` $G_f < 0$ `for` $f > 1 \Rightarrow$ `a 'runaway' that is so unstable.`

To wrap up, from the nonseasonal, the seasonal analogue of PFS is created by time-dependent equivalent of (4):

$$A + BT(x) - \frac{d}{dx}\left(k\left(1-x^2\right)\frac{dT(x)}{dx}\right) - S(1-\alpha)\xi(x) = W(x)\frac{dT}{dt}$$

`where` $t :=$ `time,` $W(x) :=$ `energy storage change, varing with latitude mainly, as fraction of land and ocean varies.`

Thus, defining $W_n = W(x_n)$ and using centered time differencing as above, obtain the following (unconditionally stable) implicit trapezoidal scheme, in which the left hand side of the equation is approximated with a backward time difference, and the right hand side is approximated with the average of the central difference scheme, evaluated at the current and the previous time step (although this itself is not simply an average since the scheme is implicitly dependent on the solution as shown above). Thus:

$$W_n \frac{T_n^{(l)} - T_n^{(l-1)}}{\Delta t} = U_n^{(l)} - \sum_m \frac{V_{nm}}{2}\left(T_m^{(l)} + T_m^{(l-1)}\right)$$

where superscript indices (l) and $(l-1)$ identify time instances assumed to be increments in constant time step Δt, and a time instance is added onto $U_n^{(l)}$ because it contains temperature-dependent terms of the E_S parameterization.

Multiplying by $2\Delta t$ and gathering terms that multiply the unknown temperatures $T_n^{(l)}$ to the left gives:

$$\sum_m (2W_n\delta_{nm} + \Delta t V_{nm})\, T_m^{(l)} = \\ = U_n^{(l)} + \sum_m (2W_n\delta_{nm} - \Delta t V_{nm})\, T_m^{(l-1)}$$

where δ_{nm} is the kronecker delta given above.

Making this less daunting by defining intermediaries:

$$H_{nm} \equiv 2W_n\delta_{nm} + \Delta t V_{nm}$$
$$X_{nm} \equiv 2W_n\delta_{nm} - \Delta t V_{nm}$$
$$Y_n^{(l)} \equiv 2\Delta t U_n^{(l)} + \sum_m X_{nm} T_m^{(l-1)}$$

implies: $\sum_m H_{nm} T_m^{(l)} = Y_n^{(l)}$.

Thus, perturbing by varying $\xi(x,t)$ rather than just $\xi(x)$ (such that initial condition is not particularly important, only polynomial fitting of current climate), simulation is "spun up" by running through a number of identically perturbed seasonal cycles until each year appears like a prior year.

Incorporating ice spread in ocean domain and snow cover in colder climates into nonseasonal or seasonal cycle, by letting the dependence of energy-in on temperature to be nonlinear and parameterized, implies that the iteration uses successive substitution in which a guess is taken at the solution, and the guess is used in calculating energy-in and hence, calculating better temperature, such that loop is repeated until temperature no longer changes between iterations, i.e., equilibrium is reached.

To conclude, besides this naive-faceted convergence-divergence arguments, more tools or objects of study can be made from as-desired progressive near-point or near-value convergence divergence structures of infinitesimal differences or sequences of functions, observing smoothness brokeness approximation empirics, and the operations of calculus; but, once more, the raison d'être that "clinch" this exposé:

1). With only albedo feedback and nothing else changing, if injection of CO_2 into the atmosphere is stopped today after attaining an instant doubling (of concentration), the climate would still continue to climb and warm for a number of years (200 million years), then slowly return to the entire pre-CO_2 (and temperature) history in another multiple (of hundred-million) years.

2). Climate seems to follow a continuous cyclical pattern. The more understanding of the feedbacks (responses) there is, the better predictions are.

While this exposé goes to brief nondescript scribble, not intended for mere "formalized" publication, the veracious demand here constitutes important factor in a growing globe-scale warming megalomania. Not to be outdone by non-sequitur ramblings or notoriously gone wild, more energetic poor-mathematics, if nothing else: What is needed is still not changing, but more robust feedback analysis that all can express confidence in, neatly blueprinted by many or *poly* Legendre's launch.

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