Weighted Nonlinear Numerical Pricing of European Put with Dividend

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Abstract: We investigate weighted nonlinear Γ problem over numerical solution of financial markets, in particular, the pricing of European puts with dividends after an integro-differential process with a generalized Brownian motion: Modeling the Merton's jump diffusion (MJD) as an analogue of the Black-Scholes-Merton, which is a heat equation in veil, we develop a Crank-Nicolson (C-N) numerical scheme to robust Γ heuristic methods, comparing stabilities and convergences with respect to underlying analytical solutions. We establish stability of the C-N scheme by considering both the classical and symmetrical C-N equals against simple iterative scheme in nonhomogenous heat equations with unbounded domain, best of results obtained. We further investigate convergence of the stable scheme over different Γ heuristic methods.

Keywords: European put, options pricing, system of linear equations, jump diffusion, heat equation, graph topology, approximate numerical scheme, tridiagonal matrix, iteration algorithm, error analysis, numerical efficiency, stability

INTRODUCTION

In an european put *option*[1], holder (or buyer) gets to pay the writer (or seller) a *premium* in purchase of a right to sell a specified quantity of an *underlying* security for a strike (or exercise) price X *on* the maturity (expiration) date T, so that the writer is obligated to meet the contract terms by buying the security, in the event the holder decides to exercise the options. Thus, supposing as part of an income play, 'Trader A' (holder or buyer) pays 'Trader B' (writer or seller) a *premium* of \$5/share in purchase of a put contract to later sell 100,000 shares of an arbitrary stock XXX (with current market price \$55/share) for \$50/share. And, on the day of expiration, market price of XXX has fallen to \$40/share. 'Trader A' then exercises the put by buying 100 shares for \$4,000,000 from the stock

[1] i.e., *a contract* to which *the right* – not the obligation – to *exercise specified quantity* of an *underlying asset* is given *its holder*; and *the obligation* – not right – to *meet the contract terms* is given *its writer*.

market, and selling them to 'Trader B' for \$5,000,000 and 'Trader B' must buy.

Profits (i.e., the net payoffs[2]), excluding commissions, are realized as follows:

For the put buyer:

$$\begin{aligned} \text{Gross payoff} &= \max(X - S^T, 0) \\ \text{Net payoff} &= -\,premium + \max(X - S^T, 0) \end{aligned}$$

For the put seller:

$$\begin{aligned} \text{Gross payoff} &= \min(S^T - X, 0) \\ \text{Net payoff} &= premium + \min(S^T - X, 0) \end{aligned}$$

taking into account the unknown security price S at T.

Not surprisingly, buyers and sellers have different motives. Buyers hope market price of the *underlying* drops so they can sell at the exercise price, which is higher than the market price. This way, they can offset the price of the premium, and hopefully make a profit as well. Sellers, on the other hand, hope price stays the same or increases, so they can keep the premium they have collected and not have to let go of money to buy. Thus, *the bull always does the opposite of a bear*[3].

However, buyers and sellers may make arbitrage profit[4] by taking, respectively, short and long positions in over-valued and under-valued securities, based on the existing arbitrage opportunities. That is, if a given European put is set, given the exercise price X, time T to maturity, initial market price S^o, volatility (or standard deviation) σ per unit time, riskless interest rate r per unit time, the goal of a a trader is to first determine if the put is under-priced or over-priced.

The *de facto* model for European put seems to be the Black-Scholes-Merton (BSM) (Black & Scholes, 1973; Merton, 1976). But, given the less conservative nature of the market, the Black-Scholes-Merton falls too short to be a preferred model. It assumes market path is always continuous or smooth (differentiability wise), and this is a far cry from the real world. The Merton's Jump Diffusion (MJD) (Merton, 1976) on the contrary takes into account some realistic near-market phenomena such as discontinuities or jumps.

That is, in the MJD, price S of underlying security is generally assumed to be continuous-time with geometric (exponential) Brownian motion. This allows price to have lognormal distribution over a short period of time, as in classical Black-Scholes-Merton, in addition to exponential distribution in time-gaps between jumps due to added Poisson-driven process. More so, the height of each jump is proportional to underlying security's initial market price and to lognormally distributed random variable.

INTEGRO-DIFFERENTIAL MODEL

Given the dividend q paying system of dividend magnitude $\rho = \frac{q}{S}$, $\rho > 0$ or dividend per unit security price, with series of jumps, where each jump is a sudden movement in security price caused by any number of change or market factors assumed to be Poisson driven, the classical MJD integro-differential equation is given by

$$\begin{aligned} \frac{\partial \widehat{V}}{\partial t} &+ (r - \rho - \lambda K) S \frac{\partial \widehat{V}}{\partial S} \\ &+ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \widehat{V}}{\partial S^2} \\ &+ \lambda \int_0^\infty \widehat{V}(t, \eta S) g(\eta) d\eta \\ &= (r + \lambda)\ \widehat{V} \end{aligned} \tag{1}$$

where

V	$:=$	price
r	$:=$	risk-free interest rate per unit time
σ	$:=$	volatility (standard dev.) per unit time
λ	$:=$	average number of jumps per unit time
η	$:=$	average jump amplitude, % of price
K	$:=$	expected average jump amplitude, defined as

$$K := E(\eta - 1) = e^{(\eta + 0.5\delta^2)} - 1$$

where $(\eta - 1) :=$ impulse function producing a jump from S to ηS, such that

$$g(\eta) \geq 0, \qquad \int_0^\infty g(\eta) d\eta = 1 \qquad \text{a.s.}$$

and

$$g(\eta) \quad := \quad \frac{1}{\sqrt{2\pi}\delta\eta} exp\left\{ -\frac{1}{2}\Big(\frac{\ln(\eta) - m}{\delta}\Big)^2 \right\}$$

where $m :=$ average of $\ln(\eta)$, and δ is the volatility of the jump size given by $\delta := \sqrt{\text{variance of } \ln(\eta)}$.

[2] Technically, the put buyer may be in, at, or out of the money depending on $S^T <, =, or > X$, and conversely, the put seller may be in, at, or out of the money depending on $S^T >, =, or < X$, respectively.

[3] • Bull wanting to sell puts because he does not expect the underlying security to go down in price.
• Bear wanting to buy put expects underlying security to go down.
• Writer of a put will not see his profits increase as the underlying security moves in price, and the writer will never receive more than the premium he gets when the put is written.
• Writing puts is an income play for the bull.

[4] If two or more securities are mispriced relative to one another.

The solution $\widehat{p}\big(\widehat{V}(t,S)\big)$ as a function of time and the security price for any put, given the underlying security in dividend q-paying jump diffusion system, is uniquely specified by a "boundary condition":

$$\widehat{p} \quad := \quad \max(X - S^o e^{-q\tau}, 0) \quad \text{ at maturity } t = t_N \quad (2)$$

where $\tau = t_N - t_0$, S^o := security's initial market price (at $t = t_0$), and X := strike price[5]. And, the resulting analytical solution of the MJD put then is given by:

$$\widehat{p}(\tau, S^o, q, \widehat{\lambda}, \widehat{r}, \widehat{\sigma}) \quad := \quad \sum_{n=0}^{\infty} \frac{e^{-\widehat{\lambda}\tau}(\widehat{\lambda}\tau)^n}{n!} p(\tau, S^o, q, \widehat{\lambda}, \widehat{r}, \widehat{\sigma})$$

$$= \sum_{n=0}^{\infty} \exp\left(-\widehat{\lambda}\tau + n\ln(\widehat{\lambda}\tau) - \sum_{i=1}^{n} \ln i\right) p(\tau, S^o, q, \widehat{\lambda}, \widehat{r}, \widehat{\sigma}) \quad (3)$$

where

$$\widehat{\lambda} = \lambda(1+K)$$

$$\widehat{\sigma}^2 = \sigma^2 + \frac{\eta\delta^2}{\tau}$$

$$\widehat{r} = r - \lambda K + \frac{n\ln(1+K)}{\tau}$$

and $p(\tau, S^o, q, \widehat{\lambda}, \widehat{r}, \widehat{\sigma})$ is the analogue[6] of analytical solution of the differential equation for the BSM (that is, a dividend q-paying system without jump):

$$\frac{\partial V}{\partial t} + (r-\rho)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial t^2} = rV \quad (4)$$

given by:

$$p(\tau, S^o, q, r, \sigma) \quad := \quad -S^o e^{-q\tau} N(-d_1) \quad + \quad X e^{-r\tau} N(-d_2) \quad (5)$$

where

$$N(x) := \text{ standard normal distribution } \mathcal{N}(0,1) \text{ CDF}$$

$$d_1 := \frac{\ln(S^o/X) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \quad (6)$$

$$d_2 := \frac{\ln(S^o/X) + (r - q - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau} \quad (7)$$

with corresponding call price:

$$c(\tau, S^o, q, r, \sigma) \quad := \quad S^o e^{-q\tau} N(d_1) \quad - \quad X e^{-r\tau} N(d_2)$$

and put-call parity:

$$p \quad := \quad c \quad + \quad X e^{-r\tau} \quad - \quad S^o e^{-q\tau}$$

[5] Note that the boundary condition for a corresponding *European call* $\widehat{c}$ options is $\widehat{c} = \max(S^o e^{-q\tau} - X, 0)$, at $t = t_N$.

[6] Note that the MJD $(\widehat{V})$ for a Derivative of an underlying security in a dividend-q jump diffusion system is

$$\widehat{V}(\tau, S^o, q, \widehat{\lambda}, \widehat{r}, \widehat{\sigma}) = \sum_{n=0}^{\infty} \frac{e^{-\widehat{\lambda}\tau}(\widehat{\lambda}\tau)^n}{n!} V(\tau, S^o, q, \widehat{\lambda}, \widehat{r}, \widehat{\sigma})$$

HEAT EQUATION

Using change of variables:

$$x := \ln\left(\tfrac{S}{X}\right), \ \tau := \tfrac{\sigma^2}{2}t, \ u(x,\tau) := \tfrac{V(S,t)}{X} e^{\alpha x + \beta\tau}$$

$$\alpha := \tfrac{r-\rho-\frac{\sigma^2}{2}}{\sigma^2}, \ \beta := \tfrac{2(r-\rho)}{\sigma^2} + \alpha^2$$

(4) becomes:

$$u_t = u_{xx} \quad (8)$$

with the boundary condition:

$$u(x,0) = \frac{\max(Xe^x - X, 0)}{X} e^{\alpha x} = \max(e^x - 1, 0)e^{\alpha x}$$

NUMERICAL APPROXIMATION

Discretization

The heat equation is discretized with respect to time and underlying security price as follows: Dividing the (S,t) plane into nicely dense grid or mesh, and approximate infinitesimal steps ΔS and Δt by a small fixed finite steps, the array of $N+1$ equally spaced grid points $t_0, t_1, t_2, \cdots, t_N$ forms an equal partition of derivative[7] time, so that for a given time to maturity τ: $t_0 = 0, t_N = \tau$, and

$$\Delta t = \frac{\tau}{N} = t_{n+1} - t_n \quad \text{ for } n = 0, 1, \cdots, N-1$$

Assuming security price cannot go below 0 and fixing $S_{\max} := S^o$, the array of $M+1$ equally spaced grid points $S_0, S_1, S_2, \cdots, S_M$ forms an equal partition of derivative security price, so that for a given security price S^o: $S_0 = 0, S_M = S^o$, and

$$\Delta S = \frac{S^o}{M} = S_{m+1} - S_m \quad \text{ for } m = 0, 1, \cdots, N-1$$

This implies the rectangular region with sides $(0, S_{\max})$ and $(0, T)$ on (S,t) is the plane of computable solution for discrete points (n,m). Equivalently, the security solution at each given point in time lies in the $(M+1)\times(N+1)$ grid. Point (n,m) on the grid is the point which corresponds to $n\Delta t$ for $n = 0, 1, \cdots, N$, with security price $m\Delta S$ for $m = 0, 1, \cdots, M$. And, the value of the derivative at time step t_n with underlying security S_m then is given by

$$\widehat{p}_{n,m} := \widehat{p}(t_n, S_m) = \widehat{p}(n\Delta t, m\Delta S) = \sum_{n=0}^{\infty} \frac{e^{-\widehat{\lambda}t_n}(\widehat{\lambda}t_n)^n}{n!} p(t_n, S_m, q, \widehat{\lambda}, \widehat{r}, \widehat{\sigma})$$

where n and m are the number of discrete increments in the time to maturity and security price, respectively, while Δt and ΔS are discrete increments of the time to maturity and the security price, respectively. That is, let $\widehat{p}_n := \widehat{p}_{n,0}, \widehat{p}_{n,1}, \cdots, \widehat{p}_{n,m}$, $n = 0, 1, 2, \cdots, N$, then $\widehat{p}_{N,m}$ is boundary value at $t = T$, for $m = 0, 1, 2, \cdots, M$.

[7] Here, small "d" derivative refers to object of differentiation, unlike big "D" Derivative, which refers to financial instrument.

Numerical Scheme

Define:

$$u(x_{right},\,\tau) \quad = \quad 0$$

and

$$u(x_{left},\,\tau) \quad = \quad \left(e^{\frac{-2rt}{\sigma^2}} \quad - \quad e^{x_{left}}\right)\left(e^{\alpha x_{left}+\beta\tau}\right)$$

which are basically when the options is considered certain to expire out of the money, and when it is considered certain to expire in the money, respectively.

From (8)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Classical Crank-Nicolson (C-N) is developed as follows:

$$\frac{u(x,\tau) - u(x,\tau-\Delta\tau)}{\Delta\tau} =$$

$$= \frac{1}{2}\frac{u(x-\Delta x,\tau) - 2u(x,\tau) + u(x+\Delta x,\tau)}{\Delta x^2}$$

$$+ \frac{1}{2}\frac{u(x-\Delta x,\tau-\Delta\tau) - 2u(x,\tau-\Delta\tau) + u(x+\Delta x,\tau-\Delta\tau)}{\Delta x^2}$$

Let $h := \tau/N_\tau := \Delta\tau$, for $N_\tau :=$ number of time steps.

Imposing $\Delta x := 1/\sqrt{rh}$ then for optimum results:

$$-\frac{rh}{2}u(x-\Delta x,\tau) + (1+rh)u(x,\tau) - \frac{rh}{2}u(x+\Delta x,\tau) \quad =$$

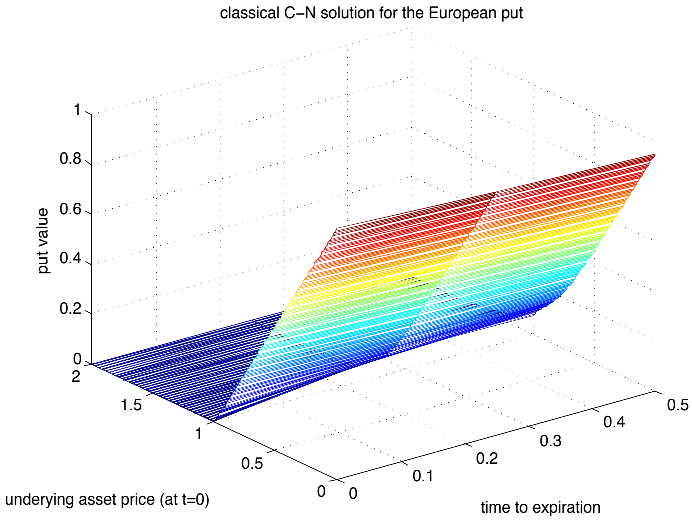
$$\frac{rh}{2}u(x-\Delta x,\tau-\Delta\tau) + (1-rh)u(x,\tau-\Delta\tau) \quad +$$

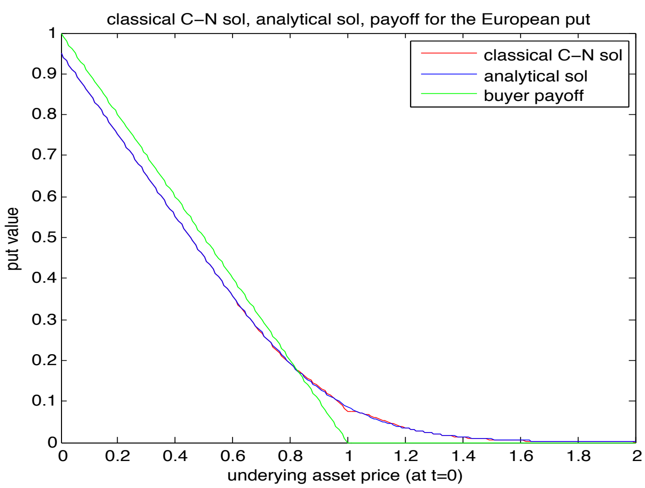
$$\frac{rh}{2}u(x+\Delta x,\tau-\Delta\tau)$$

Expressing the system in the form $Au(\tau) = b(\tau)$ then

$$A = \begin{pmatrix} 1+rh & -\frac{rh}{2} & 0 & \cdots & 0 \\ -\frac{rh}{2} & 1+rh & -\frac{rh}{2} & & \vdots \\ 0 & -\frac{rh}{2} & 1+rh & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\frac{rh}{2} \\ 0 & \cdots & 0 & -\frac{rh}{2} & 1+rh \end{pmatrix}$$

$$b(\tau) = \begin{pmatrix} \frac{rh}{2}u(x_{left},\tau-\Delta\tau) + (1-rh)u(x_{left}+\Delta x,\tau-\Delta\tau) + \frac{rh}{2}u(x_{left}+2\Delta x,\tau-\Delta\tau) + \frac{rh}{2}u(x_{left},\tau) \\ \frac{rh}{2}u(x_{left}+\Delta x,\tau-\Delta\tau) + (1-rh)u(x_{left}+2\Delta x,\tau-\Delta\tau) + \frac{rh}{2}u(x_{left}+2\Delta x,\tau-\Delta\tau) + \frac{rh}{2}u(x_{left}+3\Delta x,\tau-\Delta\tau) \\ \vdots \\ \frac{rh}{2}u(x_{right}-3\Delta x,\tau-\Delta\tau) + (1-rh)u(x_{right}-2\Delta x,\tau-\Delta\tau) + \frac{rh}{2}u(x_{right}-\Delta x,\tau-\Delta\tau)\frac{rh}{2}u(x_{left}+3\Delta x,\tau-\Delta\tau) \\ \frac{rh}{2}u(x_{right}-2\Delta x,\tau-\Delta\tau) + (1-rh)u(x_{right}-\Delta x,\tau-\Delta\tau) + \frac{rh}{2}u(x_{right},\tau-\Delta\tau) + \frac{rh}{2}u(x_{left}+3\Delta x,\tau-\Delta\tau) \end{pmatrix}$$





Given the observation, rather than discussing actual incurred truncation error, which is minimal, we more specifically focus on the stability of the solution, relatively to other[8] state-solutions.

For a nonhomogenous heat equation, in an unbounded domain, given by:

$$\begin{aligned} u_t &= u_{xx} & x \in [0,\,1],\, t \geq 0 \\ u(x,0) &= \sin(\pi x) & x \in [0,\,1] \text{ (initial condition)} \\ u(0,t) &= u(1,t) & t \geq 0 \text{ (boundary condition)} \end{aligned}$$

for $t \in [0, 0.3]$, $\Delta t := h^2 r$, x-dimension step-size h, and variable stability factor r, with analytical solution

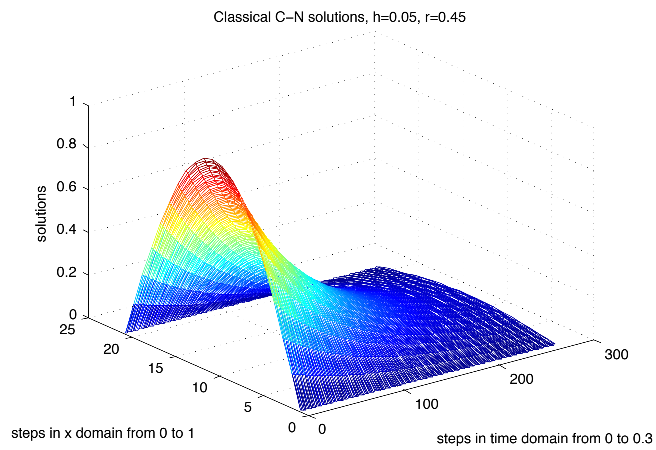
$$u(x,t) = \sin(\pi x) e^{\pi^2 t} \tag{9}$$

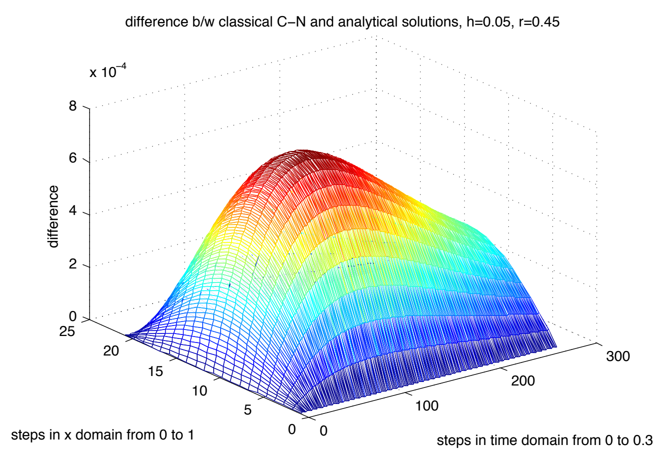
1). Classical C-N scheme gives the following results:

[8]it is part of the usuals to choose rather elementary models (although intuitive) to investigate the behavior of small errors

	$\tau = 1$	$\tau = 41$	$\tau = 81$	$\tau = 121$
n=1	0	0	0	0
n=5	0	3.4244e-04	4.3947e-04	4.2299e-04
n=9	0	5.5408e-04	7.1107e-04	6.8441e-04
n=13	0	5.5408e-04	7.1107e-04	6.8441e-04
n=17	0	3.4244e-04	4.3947e-04	4.2299e-04
n=21	0	7.8546e-17	5.0378e-17	3.2312e-17

Table 1: Table of Errors in Classical C-N *vs.* Analytical Solution, $h = 0.05$, $r = 0.45$, $N_\tau := n - 1$

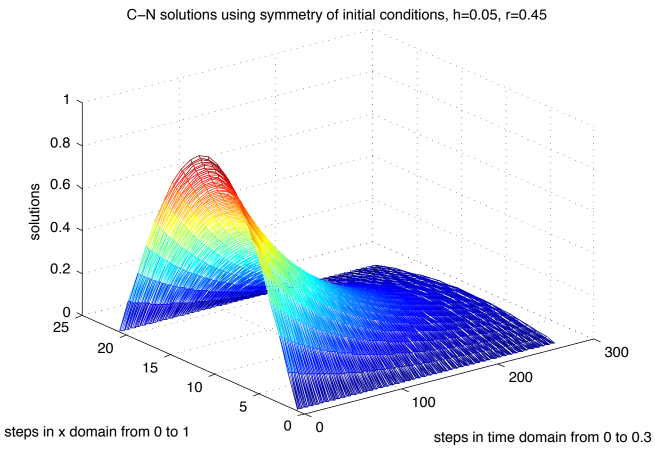


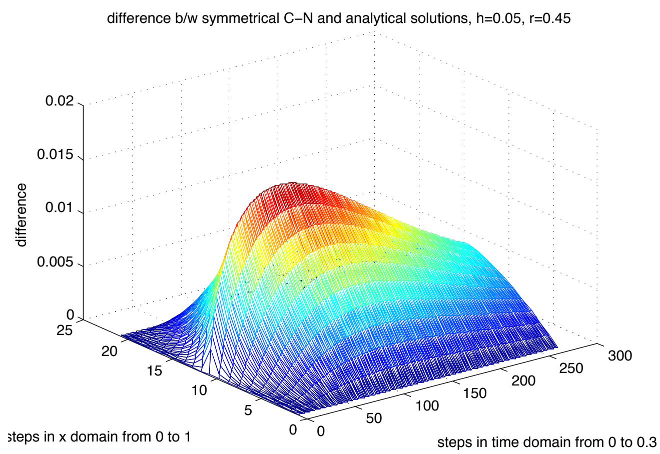


2). Symmetrical C-N scheme, which exploits the symmetry property of the initial conditions, gives the following result:

	$\tau = 1$	$\tau = 41$	$\tau = 81$	$\tau = 121$
n=1	0	0	0	0
n=5	0	3.5116e-03	5.9376e-03	6.2434e-03
n=9	0	1.0035e-02	1.2566e-02	1.2031e-02
n=13	0	1.0035e-02	1.2566e-02	1.2031e-02
n=17	0	3.5116e-03	5.9376e-03	6.2434e-03
n=21	0	7.8546e-17	5.0378e-17	3.2312e-17

Table 2: Table of Errors in Symmetrical C-N *vs.* Analytical Solution, $h = 0.05$, $r = 0.45$, $N_\tau := n - 1$





3). With simple explicit iteration scheme of the form:

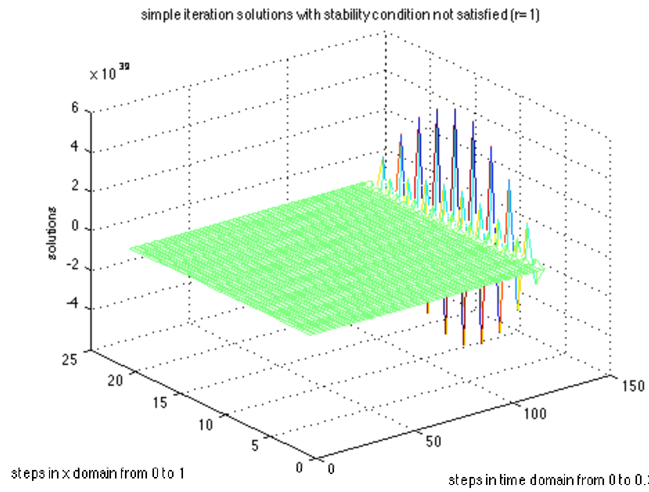
$$u_{n,i+1} = (1-2r)u_{n,i} + r(u_{n+1,i} + u_{n-1,i})$$

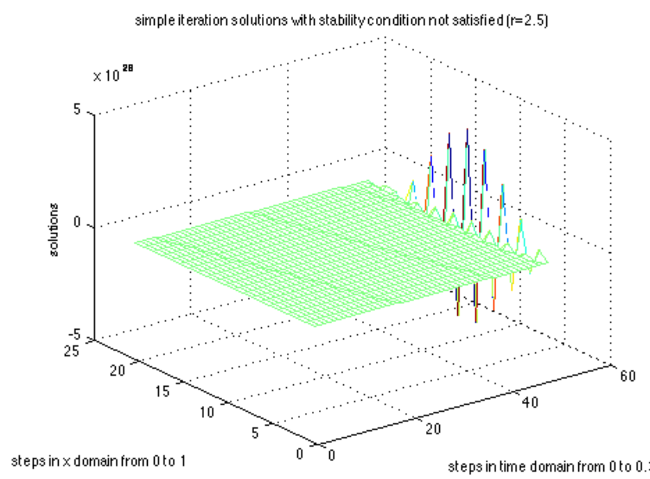
where $r := \frac{k}{h^2} \leq \frac{1}{2}$ for some k, the following result is obtained:

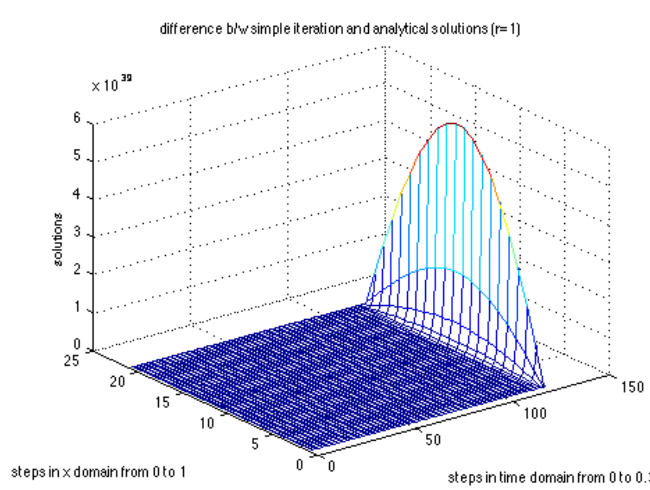
	$\tau = 1$	$\tau = 41$	$\tau = 81$	$\tau = 121$
n=1	0	0	0	0
n=5	0	7.3727e-04	6.7688e-04	4.6609e-04
n=9	0	1.1929e-03	1.0952e-03	7.5414e-04
n=13	0	1.1929e-03	1.0952e-03	7.5414e-04
n=17	0	7.3727e-04	6.7688e-04	4.6609e-04
n=21	0	5.6294e-17	2.5877e-17	1.1895e-17

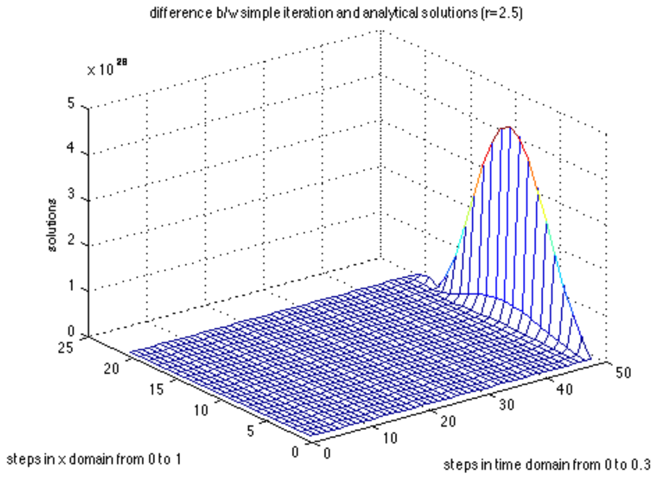
Table 3: Table of Errors in Simple Iteration *vs*. Analytical Solution, $h = 0.05$, $r = 0.45$, $N_\tau := n - 1$

Representative plots corresponding to when stability condition is not satisfied (i.e., $r > \frac{1}{2}$, e.g. $r = 1$ or 2.5) are shown below:









Thus, with the simple explicit iteration scheme, not satisfying the stability condition $r \leq 0.5$ causes the solution to diverge and only after a few iterations can the solution be, somewhat likely, completely different. And, on the one hand, h has to be very small, on the one hand, k has to be at least smaller than half the square root of h. For instance, if $h = 0.5$, k could be 0.005. The C-N scheme, is however, unconditionally stable. In particular, this is because the C-N scheme assigns equal weight to two approximations, namely: implicit and explicit, and writes the PDE at point $u(x, t-\Delta\tau/2)$ using linear interpolations at τ for all derivatives, and involves all 6 points: $u(x-\Delta x, \tau-\Delta\tau)$, $u(x, \tau-\Delta\tau)$, and $u(x+\Delta x, \tau-\Delta\tau)$ (all known), alongside $u(x-\Delta x, \tau)$,u(x, τ), u(x+$\Delta x, \tau$) (all unknown); thus, averaging both the forward and backward Euler schemes, in effect.

CONVERGENCE AND GRID REFINEMENT

It is interesting to compare numerical quality of decisions (for different processes and for different conditions); and, to compare the selecting of subset of methods that are characterized by high convergence rate and allows for getting decisions of best quality for a set of minimum computing time costs. For most promising methods, the optimal decisions of well known discrete problem of getting shortest path in graph Γ for squared time $O(n^2)$, using an algorithm that makes it convenient to compare quality of decisions for different heuristic methods with known optimum, are sufficient. Fixed number of iterations are required for quality of decisions on heuristic methods in samples of random Γs, implementing the corresponding series of computing measures over volunteer distributed computing networks for program testing of the above numerical scheme and modifications up to meta-empirical optimization procedures (performed in automated mode with tens of hours of simulated computing time). The C-N scheme overall is far less computationally intensive in the demand heuristics, requiring much less number of cycles for decisions on denser topology space-time, in comparison with simpler methods.

CONCLUSION

With the experimental data analytics, a set of conclusion can be formulated. First, for high density Γ, well known heuristic methods without parameter modifications provide promising numerical quality of decisions, with the most promising of popular methods being ant-colony optimization method and genetic method. The drawback of the C-N scheme on MJD pricing of dividend European put, compared to other implicit or explicit schemes which tend to give error in opposite directions, is due to the fact that: In the MJD, continuous-time lognormal diffusion path and the Poisson-driven jump process are assumed to be mutually independent, hence, heuristic methods are almost-surely rather predicting, due to non-iid events in the actual return evolution. While this drawback lurks, the MJD C-N scheme is yet to be investigated in energy market, electricity derivatives, and other spot markets. Besides, an investigation which parallels the heat equation form, for convergence and other numerical behaviors, could be done in other representative PDE forms such as:

1). Fokker-Planck equation (in 1-Dimension):

$$L := \frac{\partial u}{\partial t} + \frac{\partial}{\partial t}[A(t,x)u] - \frac{1}{2}\frac{\partial^2}{\partial x^2}[B(t,x)u] = 0$$

where $A(t,x)$ and $B(t,x)$ are smooth functions representing coefficients of drift and diffusion respectively; $u = u(t,x)$ is probability density; the initial condition is a Dirac delta function in $x = 1$, and the distribution drifts are toward $x = 0$.

2). Burgers equation:

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = v\frac{\partial^2 w}{\partial x^2} \qquad \text{where } w := w(x,t).$$

3). Schrödinger's equation, which appears in varied ways.

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