Quasicontractivity Law for Conformal Submodule

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Abstract

We consider boson-fermion conservative, symmetric submodule $\mathbb{Q}S_{\xi,\ell}$, $1 \ll \ell \leqslant \xi$, for i.i.d, d-dimension homogenous quartets, toroid-convex collision, Maxwell-Galiliean isotropic volume, Markov $\mathbb{Q}F_{\xi,\ell}$ scattering. Under finite moments, scalar speeds of velocity distributions, we get a finite-order power law of quasicontractivity coupling estimation, for d-dimensional N-particle in sphere-convex toroidal collision system, on conformal submodule $\mathbb{Q}F_{\xi,\ell}$.

Keywords: conformal-submodule, Markov-quasicontractivity

For $X \in \mathbb{Q}S^d$ iid uncoupled d-dimension N-particle system of homogeneous collisions on convex toroid sphere, particle symmetries S_N Markov process

$$t \longrightarrow X_t^N := (X_{t,(1)}^N, \ldots, X_{t,(N)}^N) \in \left(\mathbb{Q}S^d\right)^N \sim (\mathrm{S}_N) \qquad (1)$$

satisfies conservation laws, if $\forall\, t \geqslant 0$:

$$\mathbb{E}\left(X_t^N\right)_N = 0 \quad \text{a.s.}, \qquad \mathbb{E}\left(\left|X_t^N\right|^2\right)_N = 1 \quad \text{a.s.} \qquad (2)$$

where $\mathbb{E}\left(\cdot\right)_N \equiv \frac{1}{N}\sum_{\xi=1}^N$. Moreover, the process is conformal covariance reversible, with invariant uniform distribution

$$\mathrm{U}_{0,1}^N \stackrel{\text{def}}{=} \mathrm{U}\left\{X \in \left(\mathbb{Q}S^d\right)^N \;\middle|\; \text{(2) holds}\right\};$$

equivalently, a Riemannian volume $\mathrm{U}_{0,1}^N$ of associated collision toroid sphere, or conditional distribution $\mathrm{U}_{0,1}^N$ of the average momentum energy observables on the collision toroid sphere.

The Markov dynamics of (1) is specified by random coupled collisions of (Levy) jump type

$$\chi \stackrel{\mathrm{def}}{=} \int_{[0,\pi]} \sin^2\theta \, b(\mathrm{d}\theta) \quad < \quad +\infty \tag{3}$$

satisfying: Momentum, K.E. (iid mass) couple conservation; constant collision rates (i.e. Maxwell); isotropic random step on Euclidean-sphere collision directions (i.e. Galilean); particle-pair velocity-difference direction scattering (deviation) angle $\theta \in [0, \pi]$, in kernel $b(d\theta)$, with positive Levy measure on $[0, \pi]$ scattering angle random steps.

Now, we say the probabilistic coupling of two copies of a Markov process which in itself is again Markov, denoted

$$\begin{aligned} t \longrightarrow \left({X^{(i)}}_t^N , \, {X_t^{(j)}}^N \right) \equiv \\ \equiv \left(\left(X^{(i)}_{t,(1)} , X^{(j)}_{t,(1)} \right) , \ldots , \left(X^{(i)}_{t,(N)} , X^{(j)}_{t,(N)} \right) \right) \in \left(\mathbb{Q}F^d \times \mathbb{Q}F^d \right)^N \end{aligned} \quad . \tag{4}$$

The later (4) is *globally* invariant on toric sphere by particle permutation, such that ${X^{(i)}}_t^N \in \left(\mathbb{Q}F^d\right)^N$ and ${X^{(j)}}_t^N \in \left(\mathbb{Q}F^d\right)^N$ indpendently satisfies conservation laws (2), and collisions are coupled by the rules:

(i) Collision-times and collisional-particles are the same

(ii) Scattering angles are the same

(iii) Isotropic random step on collision-direction is coupled using plain parallel transport (in geometric sense), with no reflexion.

With the sphere being a strictly positively curved manifold, the latter coupling is bound to be almost surely decreasing, in the sense that for any initial condition and $0 \leqslant t \leqslant t+h$,

$$\mathbb{E}\left(\left|{X^{(i)}}_{t+h}^N - {X^{(j)}}_{t+h}^N\right|^2\right)_N \leqslant \mathbb{E}\left(\left|{X^{(i)}}_t^N - {X^{(j)}}_t^N\right|^2\right)_N \quad \text{a.s.} \tag{5}$$

and, for uniform average coupling distance-time derivative,

$$\frac{\mathsf{d}}{\mathsf{dt}}\mathbb{E}\left(\left|{X^{(i)}}_t^N - {X^{(j)}}_t^N\right|^2\right)_N = -\mathbb{E}\left(\mathcal{C}\left({X^{(i)}}_t^N, {X^{(j)}}_t^N, {X^{(i)}}_{*,t}^N, {X^{(j)}}_{*,t}^N\right)\right)_N \leqslant 0. \tag{6}$$

Then for all velocity-difference couple $x^{(j)}-x_*^{(j)},\ x^{(i)}-x_*^{(i)}$ alignment,

$$
\begin{aligned}
&\mathcal{C}(x^{(i)},x^{(j)},x_*^{(i)},x_*^{(j)}) \ = \\
&\quad = \ \lambda\,\tfrac{c_{d-1}}{c_{d-3}}\left|x^{(i)}-x_*^{(i)}\right|\left|x^{(j)}-x_*^{(j)}\right| - (x^{(i)}-x_*^{(i)})\cdot(x^{(j)}-x_*^{(j)}) \ \geqslant \ 0
\end{aligned}
\tag{7}
$$

where $c_d = \displaystyle\int_0^{\pi/2} \sin^d(\varphi)\,d\varphi$ denotes d'th Wallis integral; $\lambda > 0$ is variant of scattering angle kernel in (3); i.e. for coupling centered normalized variables $\mathbb{Q}F^d$, the original generally sharp inequality is given by:

$$
\begin{aligned}
f\left(\mathbb{E}\left(\left|{x^{(i)}}^N - {x^{(j)}}^N\right|^2\right)_N\right) \ \leqslant \ \min&\left(\kappa_{\mathbb{E}\left({x^{(i)}}^N\otimes {x^{(i)}}^N\right)_N},\ \kappa_{\mathbb{E}\left({x^{(i)}}^N\otimes {x^{(i)}}^N\right)_N}\right) \\
\times \ \mathbb{E}&\left(\begin{array}{c} \left|{x^{(i)}}^N - {x_*^{(i)}}^N\right|^2 \left|{x^{(j)}}^N - {x_*^{(j)}}^N\right|^2 \\ -\left(\left({x^{(i)}}^N - {x_*^{(i)}}^N\right)^2 \cdot \left({x^{(j)}}^N - {x_*^{(j)}}^N\right)\right)^2 \end{array}\right)_N
\end{aligned}
\tag{8}
$$

such that vectors ${x^{(i)}}^N, {x^{(j)}}^N \in (\mathbb{Q}F^d)^N$ satisfy conservation (2).

Moreover, the centered normalized coupling, *Cauchy* number

$$\kappa_S \stackrel{\text{def}}{=} (1 - |||S|||)^{-1} \in [\frac{d}{d-1}, +\infty] \tag{9}$$

is function of spectral radius $|||S||| \leqslant 1$ positive-trace 1 symmetric matrix, by

$$f\colon [0,4] \longrightarrow [0,1] \;\Big|\; q \longmapsto q - \frac{q^2}{4} \tag{10}$$

i.e. positive concave function for all $f(q) \underset{q \longrightarrow 0}{\sim} q \mid f(4-q) = f(q)$ ensuring symmetry ${x^{(j)}}^N \longrightarrow -{x^{(j)}}^N$ in (8) where equality is satisfied under the two sufficient conditions:

(i) Co-linearity of $\frac{x^{(i)}_{(n)}}{\left|x^{(i)}_{(n)}\right|}$ and $\frac{x^{(j)}_{(n)}}{\left|x^{(j)}_{(n)}\right|}$, $\forall\ 1 \leqslant n \leqslant N$

(ii) Isotropy of co-variances

$\mathbb{E}\Big({x^{(i)}}^N \otimes {x^{(i)}}^N\Big)_N = \mathbb{E}\Big({x^{(i)}}^N \otimes {x^{(j)}}^N\Big)_N = \frac{1}{d}\mathrm{Id}$

or $\mathbb{E}\Big({x^{(j)}}^N \otimes {x^{(j)}}^N\Big)_N = \mathbb{E}\Big({x^{(i)}}^N \otimes {x^{(j)}}^N\Big)_N = \frac{1}{d}\mathrm{Id}.$

Comparing alignment functional in (8) RHS (sharp upper bound of square coupling distance) with coupling creation functional (7), the difference is weight of form $|x-x_*|\,|y-y_*|$ forbidding all strong "coupling/coupling creation" inequality

$$\frac{f\left(\mathbb{E}\left(\left|{x^{(i)}}^N-{x^{(j)}}^N\right|^2\right)_N\right)}{\mathbb{E}\left(\beta\left({x^{(i)}}^N,{x^{(j)}}^N,{x^{(i)}}^N_*,{x^{(i)}}^N_*\right)\right)_N}\leqslant r<+\infty\,;$$

$$\mathbb{E}\left(\beta\left({x^{(i)}}^N,{x^{(j)}}^N,{x^{(i)}}^N_*,{x^{(i)}}^N_*\right)\right)_N\overset{\text{def}}{=}\frac{1}{N^2}\sum_{n_1,n_2=1}^{N}\beta\left({x^{(i)}}^N_{(n_1)},{x^{(j)}}^N_{(n_1)},{x^{(i)}}^N_{(n_2)},{x^{(j)}}^N_{(n_2)}\right)$$

for universal constant $r>0$ independent of N, where pair $({x^{(i)}}^N,{x^{(j)}}^N)\in(\mathbb{Q}S^d\times\mathbb{Q}S^d)^N$ satisfy conservation (2), i.e. Hölder inequality, $\forall\,\varepsilon\in\,]0,+\infty[$ directly yields the weaker power law:

$$\frac{f\left(\mathbb{E}\left(\left|{x^{(i)}}^N-{x^{(j)}}^N\right|^2\right)_N\right)}{\mathbb{E}\left(\beta\left({x^{(i)}}^N,{x^{(j)}}^N,{x^{(i)}}^N_*,{x^{(i)}}^N_*\right)\right)_N^{\frac{\varepsilon}{1+\varepsilon}}}\leqslant r_{\varepsilon,{x^{(i)}}^N,{x^{(j)}}^N}<+\infty. \qquad (11)$$

Conclusion:

The inequality (11) in form of order ε power law gives estimation of coupling quasicontractivity on scalar speed

$$\underset{t \;\longrightarrow\; +\infty}{\sim} \; t^{-\varepsilon}$$

with the RHS

$$r_{\varepsilon,\, x^N,\, y^N}$$

controllable by N-average, finite order $> 2 + \varepsilon$ moment velocity distribution.

Result was stirred by: Cercignani (1982); Carlen & Carvalho (1992); Bobylev & Cercignani (1999); Carlen, Gabetta, & Toscani (1999); and, Toscani & Villani (1999); all in so-called method of "entropy" inequalities. However, analysis here is independent of N, allowing in convex toroid sphere collision, a fully conformal invariance driven only by the final Hölder inequality (11), a generality in simplicity which is the main motivation for this work.

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Thank you!