Lie Algebra Weight System

Matthew Bernard

In: Combinatorics of Vassiliev invariants
Moscow, Russia

December 11, 2014

Abstract

We prove construction of weight system ω_s with respect to Lie algebras $\mathfrak{sl}_2$, $\mathfrak{gl}_N$ by 4-term relations in Vasiliev knot invariants; applications include Knizhnik – Zamolodchikov – Kontsevich (KZK) and the Knizhnik – Zamolodchikov – Bernard (KZB) weight systems for knot invariant: a constant characteristic of an arbitrary set of knots which is independent of representation (although the set is not without a well defined representation) with respect to ambient isotopy.

Lie alg wt sys for the algebra $\mathcal{A}^{fr}$

1.1 Universal Lie algebra weight systems

Following Kontsevich [1] with the basic ideas in [2]: Let $\mathfrak{g}$ be a metrized Lie algebra over $\mathbb{R}$ or $\mathbb{C}$, that is, a Lie algebra with an ad-invariant non-degenerate bilinear form $\langle \cdot, \cdot \rangle$:

$$\beta \ : \ \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$$

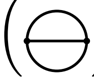
$$\beta([X,Y],Z) \ = \ -\beta(Y,[X,Z]), \ \ \forall\, X,Y,Z \in \mathfrak{g}$$

$\omega_s =$ trace of operator for finite representation of $\mathfrak{g}$
$\omega_s \in ZU(\mathfrak{g}) =$ center of universal enveloping algebra $U(\mathfrak{g})$.

1.1.1 Remark. $\omega_s =$ symbols of quantum group invariants (since quantum invariant is polynomial in q and q^{-1}, and for $q = e^h$, coefficient of h^n can be considered in Taylor expansion of the quantum invariant. Another way of constructing weight systems, also due to Kontsevich is using *marked surfaces*.

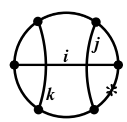
1.2 Vasiliev's Derivation of Lie Algebra Wt Sys

1.2.1 Derivation.

$$\omega_s = \varphi_{\mathfrak{g}}\Big($$  $$\Big)$$

$$= \sum_{i=1}^{m} e_i e_i^* \qquad \text{i.e., the Quadratic Casimir}$$

1.2.2 Derivation.

$$\omega_s = \varphi_{\mathfrak{g}}\Big($$ 

$$\Big)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} e_i e_j e_k e_i^* e_k^* e_j^*$$

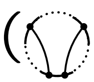
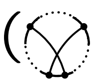
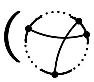
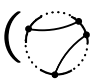
1.2.1 Theorem.

1). $\varphi_{\mathfrak{g}}(D)$ *belongs to the center* $ZU(\mathfrak{g})$ *ad-invariant subspace*

$$U(\mathfrak{g})^{\mathfrak{g}} = \{x \in U(\mathfrak{g}) \mid xy = yx, \ \forall\, y \in \mathfrak{g}\}$$

of the universal enveloping algebra $U(\mathfrak{g})$.

2). *The function* $f : D \longmapsto \varphi_{\mathfrak{g}}(D)$ *satisfies 4-term relations*

$f($$) - f($$) + f($$) - f($$) = 0.$

3). *The resulting map* $\varphi_{\mathfrak{g}} : \mathcal{A}^{fr} \to ZU(\mathfrak{g})$ *is homomorphism of algebras.*

Proof.

1). $\varphi_{\mathfrak{g}}(D)$ commutes with any basis element e_r. So, choose $e_i^* = e_i$ for all i, and expand commutator of e_r and $\varphi_{\mathfrak{g}}(D)$ into sum $2n$ of expressions, similar to $\varphi_{\mathfrak{g}}(D)$, only with one of e_i's replaced by its commutator with e_r. Concretely,

$$
\begin{aligned}
&[e_r, \sum_{ij} e_i e_j e_i e_j] \\
&= \sum_{ij} [e_r, e_i] e_j e_i e_j + \sum_{ij} e_i [e_r, e_j] e_i e_j + \sum_{ij} e_i e_j [e_r, e_i] e_j + \sum_{ij} e_i e_j e_i [e_r, e_j] \\
&= \sum_{ijk} c_{rik} e_k e_j e_i e_j + \sum_{ijk} c_{rjk} e_i e_k e_i e_j + \sum_{ijk} c_{rik} e_i e_j e_k e_j + \sum_{ijk} c_{rjk} e_i e_j e_i e_k \\
&= \sum_{ijk} c_{rik} e_k e_j e_i e_j + \sum_{ijk} c_{rjk} e_i e_k e_i e_j + \sum_{ijk} c_{rki} e_k e_j e_i e_j + \sum_{ijk} c_{rkj} e_i e_k e_i e_j .
\end{aligned}
$$

The first sum cancels with the third, and the second cancels with the fourth due to antisymmetry property of structure constants c_{ijk}.

2). Still assuming that the basis $\{e_i\}$ is $\langle\cdot,\cdot\rangle$–orthonormal, one of the pairwise differences of the chord diagrams that constitute the 4 term relation is sent by $\varphi_{\mathfrak{g}}$ to

$$\sum c_{ijk} \ldots e_i \ldots e_j \ldots e_k \ldots$$

while the other goes to

$$\sum c_{ijk} \ldots e_j \ldots e_k \ldots e_i \ldots \;=\; \sum c_{kij} \ldots e_i \ldots e_j \ldots e_k \ldots$$

where the equality is due to cyclic symmetry of structure constants c_{ijk} in orthonormal basis.

3). By the indifference wrt basis, we place the base point in the product diagram $D_1 \cdot D_2$ between D_1 and D_2. Therefore, the evident identity $\varphi_{\mathfrak{g}}(D_1 \cdot D_2) = \varphi_{\mathfrak{g}}(D_1)\varphi_{\mathfrak{g}}(D_2)$.

1.2.1 Remark.

1). If D is a diagram with n chords,

$$\varphi_{\mathfrak{g}}(D) = c^n + \{\text{terms of degree less than } 2n \text{ in } U(\mathfrak{g})\},$$

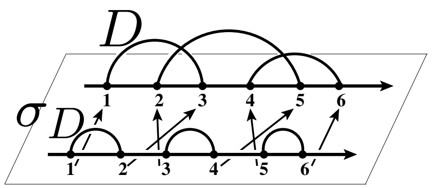
where c is the quadratic Casimir.

2). The highest degree term of $\varphi_{\mathfrak{g}}(D)$ does not depend on D, because we can permute the endpoints of chords on the circle without changing the highest term of $\varphi_{\mathfrak{g}}(D)$, and all additional summands arising as commutators have degrees less than $2n$.

3). If D is a diagram with n isolated chords, then $\varphi_{\mathfrak{g}}(D) = c^n$, i.e. the nth power of diagram with one chord.

4). The action of the center $ZU(\mathfrak{g})$ consists in taking the commutator & it's isomorphic to the algebra of polynomials in certain variables $c_1 = c, c_2, \ldots, c_r$, where $r = rank(\mathfrak{g})$.

1.2.3 Derivation. Given



$$\sigma_D = (132546)$$

Then the value of universal Lie algebra weight system $\varphi_{\mathfrak{g}}(D)$ is the image of the nth tensor power $\langle \cdot, \cdot \rangle^{\otimes n}$ under the map

$$\mathfrak{g}^{\otimes 2n} \xrightarrow{\sigma_D} \mathfrak{g}^{\otimes 2n} \to U(\mathfrak{g})$$

where the second map is the natural projection of the tensor algebra on $\mathfrak{g}$ to its universal enveloping algebra.

1.3 Universal $\mathfrak{sl}_2$ weight system

Consider the Lie algebra $\mathfrak{sl}_2$ of 2×2 matrices with zero trace i.e. the three-dimensional Lie algebra spanned by matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with commutators

$$[H,E] = 2E,\ [H,F] = -2F,\ [E,F] = H.$$

From the symmetric bilinear form property $\langle x,y\rangle = \mathrm{Tr}(xy)$:

$$\langle H,H\rangle = 2,\ \langle H,E\rangle = 0,\ \langle H,F\rangle = 0,$$
$$\langle E,E\rangle = 0,\ \langle E,F\rangle = 1,\ \langle F,F\rangle = 0$$

therefore, ad-invariant and non-degenerate.

The corresponding dual basis is given by

$$H^* = \frac{1}{2}H, \qquad E^* = F, \qquad F^* = E.$$

Hence, the Casimir element is given by $c = \frac{1}{2}HH + EF + FE$.

1.3.1 Remark.

1). The centre $ZU(\mathfrak{sl}_2)$ is isomorphic to algebra of polynomials in single variable c. The value $\varphi_{\mathfrak{sl}_2}(D)$ is thus a polynomial in c.

2). The algebra $\mathfrak{sl}_2$ is simple, hence, any invariant form is equal to $\lambda\langle\cdot,\cdot\rangle$ for some constant λ.

3). The corresponding Casimir element c_λ, as an element of the universal enveloping algebra, is related to $c = c_1$ by $c_\lambda = \frac{c}{\lambda}$.

4). Therefore, the weight system

$$\varphi_{\mathfrak{sl}_2}(D) = c^n + a_{n-1}c^{n-1} + a_{n-2}c^{n-2} + \cdots + a_2c^2 + a_1c.$$

and the corresponding weight system to $\lambda\langle\cdot,\cdot\rangle$:

$$\varphi_{\mathfrak{sl}_2,\lambda}(D) = c_\lambda^n + a_{n-1,\lambda}c_\lambda^{n-1} + a_{n-2,\lambda}c_\lambda^{n-2} + \cdots + a_{2,\lambda}c_\lambda^2 + a_{1,\lambda}c_\lambda$$

are related by $\varphi_{\mathfrak{sl}_2,\lambda}(D) = \frac{1}{\lambda^n} \cdot \varphi_{\mathfrak{sl}_2}(D)|_{c=\lambda\cdot c_\lambda}$

or

$$a_{n-1} = \lambda a_{n-1,\lambda}, a_{n-2} = \lambda^2 a_{n-2,\lambda}, \ldots, a_2 = \lambda^{n-2}a_{2,\lambda}, a_1 = \lambda^{n-1}a_{1,\lambda}.$$

1.3.1 Theorem. *Let $\varphi_{\mathfrak{sl}_2}$ = weight system for $\mathfrak{sl}_2$, with the invariant form $\langle \cdot, \cdot \rangle$. Take a chord diagram D and choose a chord a of D. Then*

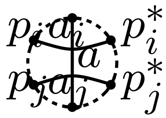
$$\varphi_{\mathfrak{sl}_2}(D) = (c-2k)\,\varphi_{\mathfrak{sl}_2}(D_a) + 2 \sum_{1\leq i<j\leq k} \Big(\varphi_{\mathfrak{sl}_2}(D^{\parallel}_{i,j}) - \varphi_{\mathfrak{sl}_2}(D^{\times}_{i,j})\Big)$$

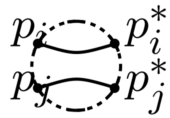
where:

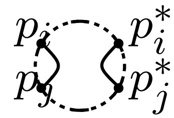
k = *number of chords that intersect the chord a*
D_a = *chord diagram obtained from D by deleting chord a.*

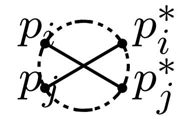
$D^{\parallel}_{i,j}$ *and* $D^{\times}_{i,j}$ *are the chord diagrams obtained from D_a by*:

Drawing diagram D so that chord a is vertical; then considering arbitrary pair of chords a_i and a_j, each different from a and intersecting a, denote by: p_i and p_j the endpoints of a_i and a_j lieing to the left of a; and, p_i^* and p_j^* the endpoints of a_i and a_j lieing to the right of a, so that there are three ways to connect the four points p_i, p_i^*, p_j, p_j^* by two chords. Then, thus:

$D =$ 

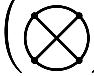
$D_a =$ 

$D^{\shortparallel}_{i,j} =$ 

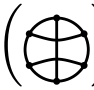
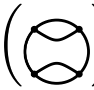
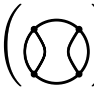
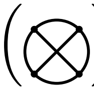
$D^{\times}_{i,j} =$ 

D_a is where those two chords are $(p_i, p_i^*), (p_j, p_j^*)$. Diagram $D^{\shortparallel}_{i,j}$ has chords $(p_i, p_j), (p_i^*, p_j^*)$; diagram $D^{\times}_{i,j}$ has $(p_i, p_j^*), (p_i^*, p_j)$; while all other chords are the same in all the diagrams.

1.3.2 Remark. The theorem allows recursive computation of $\varphi_{\mathfrak{sl}_2}(D)$, as D_a, $D^{\shortparallel}_{i,j}$, and $D^{\times}_{i,j}$, each has one chord less than D.

1.3.1 Derivation. $\varphi_{\mathfrak{sl}_2}\Big($  $\Big) = (c-2)c$. In this case, $k = 1$ and the sum in the right hand side is zero, since no pairs (i, j).

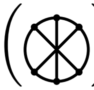
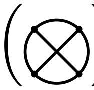
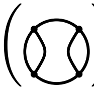
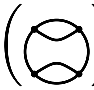
1.3.2 Derivation.

$\varphi_{\mathfrak{sl}_2}\Big($  $\Big) = (c-4)\,\varphi_{\mathfrak{sl}_2}\Big($  $\Big) + 2\,\varphi_{\mathfrak{sl}_2}\Big($  $\Big) - 2\,\varphi_{\mathfrak{sl}_2}\Big($  $\Big)$

$$= (c-4)c^2 + 2c^2 - 2(c-2)c$$

$$= (c-2)^2 c.$$

1.3.3 Derivation.

$\varphi_{\mathfrak{sl}_2}\Big($  $\Big) = (c-4)\,\varphi_{\mathfrak{sl}_2}\Big($  $\Big) + 2\,\varphi_{\mathfrak{sl}_2}\Big($  $\Big) - 2\,\varphi_{\mathfrak{sl}_2}\Big($  $\Big)$

$$= (c-4)(c-2)c + 2c^2 - 2c^2$$

$$= (c-4)(c-2)c.$$

1.3.3 Remark. Choosing invariant form $\lambda\langle\cdot,\cdot\rangle$, we obtain

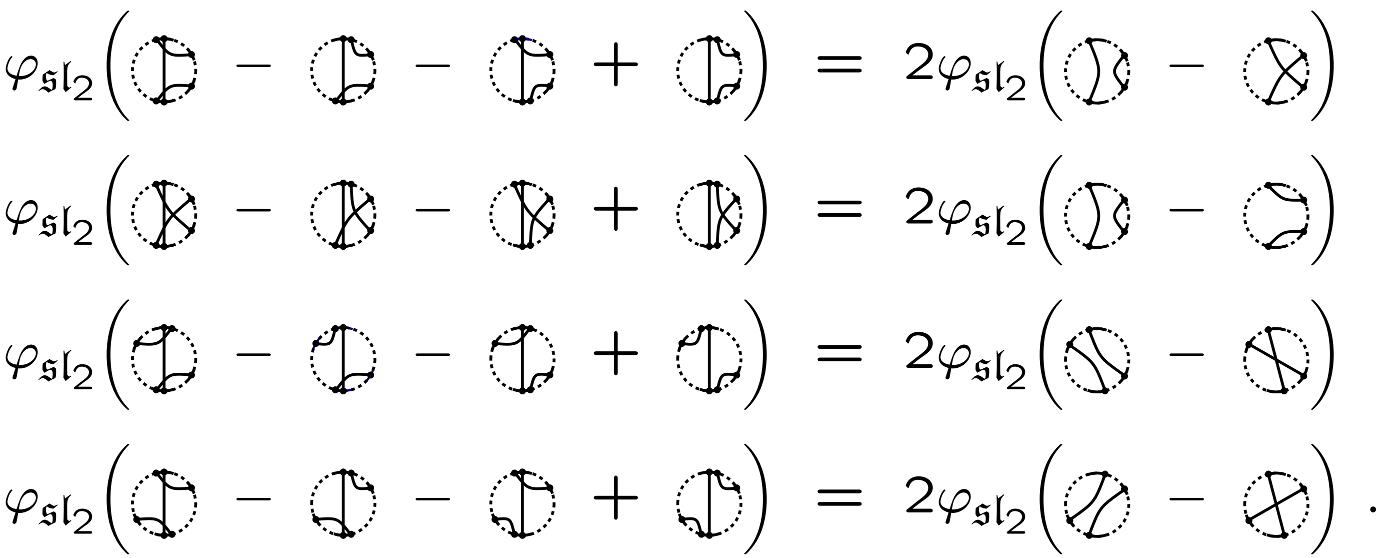
$$\varphi_{\mathfrak{sl}_2,\lambda}(D) \;=\; \Big(c_\lambda - \frac{2k}{\lambda}\Big)\varphi_{\mathfrak{sl}_2,\lambda}(D_a) + \frac{2}{\lambda}\sum_{1\le i<j\le k}\Big(\varphi_{\mathfrak{sl}_2,\lambda}(D^{\parallel}_{i,j}) \;-\; \varphi_{\mathfrak{sl}_2,\lambda}(D^{\times}_{i,j})\Big)\,.$$

If $k=1$, then the second summand vanishes. In particular, for the Killing form ($\lambda=4$) and $k=1$, then

$$\varphi_{\mathfrak{g}}(D) \;=\; (c - 1/2)\,\varphi_{\mathfrak{g}}(D_a).$$

It is interesting that the last formula is valid for any simple Lie algebra $\mathfrak{g}$ with the Killing form and any chord a which intersects precisely one other chord. A good exercise is a generalization of this fact in the case $\mathfrak{g}=\mathfrak{sl}_2$.

1.3.1 Lemma (6-term relations for universal $\mathfrak{sl}_2$ wt system)**.**
Let $\varphi_{\mathfrak{sl}_2}$ = *weight system for* $\mathfrak{sl}_2$ *with invariant form* $\langle\cdot,\cdot\rangle$*, then*



Proof. ♡.

1.3.4 Remark. These relations give recursive computation of $\varphi_{\mathfrak{sl}_2}(D)$, as right hand side chord diagrams have one chord less than left hand side. On left hand side, the last three diagrams are simpler than the first, in lesser number of intersections.

1.4 Bar-Natan representation vs. Kontsevich's

$T\colon \mathfrak{g} \longrightarrow \mathsf{End}(V)$ extends to $U(T)\colon U(\mathfrak{g}) \longrightarrow \mathsf{End}(V)$. Composition

$$\mathcal{A} \xrightarrow{\varphi_{\mathfrak{g}}} U(\mathfrak{g}) \xrightarrow{U(T)} \mathsf{End}(V) \xrightarrow{\mathsf{Tr}} \mathbb{C}$$

gives the weight system associated with the representation

$$\varphi_{\mathfrak{g}}^T = \mathsf{Tr} \circ U(T) \circ \varphi_{\mathfrak{g}}$$

where the map $\varphi_{\mathfrak{g}}^T$ is not generally multiplicative.

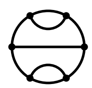
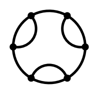
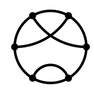
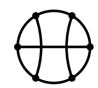
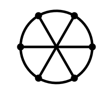
1.4.1 Remark. If T is irreducible, then, by Schur Lemma (i.e., $gV \subseteq V, \forall g \in G := GL_N(\mathbb{C})$ irreducible, i.e., $V \subseteq G$ is either $V = 0$, or $V = \mathbb{C}^n$), it implies every element of centre $ZU(\mathfrak{g})$ is represented (via $U(T)$) by a scalar operator $\mu \cdot \mathsf{id}_V$, so that its trace equals $\varphi_{\mathfrak{g}}^T(D) = \mu \dim V$, and the number $\mu = \frac{\varphi_{\mathfrak{g}}^T(D)}{\dim V}$, which is a function of the chord diagram D, is a weight system that is clearly multiplicative.

1.5 Algebra $\mathfrak{sl}_2$ with standard representation

On standard 2-dimensional representation St of $\mathfrak{sl}_2$, Casimir is

$$c = \frac{1}{2}HH + EF + FE = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix} = \frac{3}{2} \cdot \mathrm{id}_2.$$

In degree 3 we have the weight systems:

D					
$\varphi_{\mathfrak{sl}_2}(D)$	c^3	c^3	$c^2(c-2)$	$c(c-2)^2$	$c(c-2)(c-4)$
$\varphi^{St}_{\mathfrak{sl}_2}(D)$	$27/4$	$27/4$	$-9/4$	$3/4$	$15/4$
$\varphi'^{St}_{\mathfrak{sl}_2}(D)$	0	0	0	12	24

Here last row represents unframed weight system by deframing procedure on second row. It comes from fact that space $\mathcal{W}$ of unframed weight systems is dual to space $\mathcal{A}$ of unframed chord diagrams, just as space $\mathcal{W}^{fr}$ of framed weight systems is dual to space $\mathcal{A}^{fr}$ of framed chord diagrams. And, $\mathcal{A}$ is quotient of $\mathcal{A}^{fr}$ by subspace spanned by all diagrams with an isolated chord.

Moreover, in terms of multiplication in $\mathcal{A}^{fr}$, $\mathcal{A}$ is the ideal of $\mathcal{A}^{fr}$ generated by Θ, the chord diagram with one chord, thus:

$$\mathcal{A} = \mathcal{A}^{fr}/(\Theta).$$

That is, the explicit formula (to get $\varphi'^{St}_{\mathfrak{sl}_2}$ from deframing $\varphi^{St}_{\mathfrak{sl}_2}$) for linear operator p using some n procedure $p_n \colon \mathcal{A}^{fr}_n \longrightarrow \mathcal{A}^{fr}_n$ by

$$p_n(D) := \sum_{J \subseteq D} (-\Theta)^{n-|J|} \cdot D_J$$

where:

$[D]$ = set of chords in diagram D
D_J = subdiagram of D with only chords left from J

such that the sum of p_n over all n is the operator $p : \mathcal{A}^{fr} \to \mathcal{A}^{fr}$.

1.6 Algebra $\mathfrak{gl}_N$ with standard representation

Consider standard rep of Lie algebra $\mathfrak{g} = \mathfrak{gl}_N$. Fixing trace of matrices-product as ad-invariant form: $\langle x, y \rangle = \mathrm{Tr}(xy)$, with algebra of $\mathfrak{gl}_N$ being linearly spanned by matrices (e_{ij}) by $e_{ij} = 1$, $\forall\, i = j$, and 0 everywhere else. Then

$$\langle e_{ij}, e_{kl} \rangle = \delta_i^l \delta_j^k \qquad (\delta = \text{Kronecker delta}).$$

Therefore, duality between $\mathfrak{gl}_N$ and $(\mathfrak{gl}_N)^*$ defined by $\langle \cdot, \cdot \rangle$ is given by $e_{ij}^* = e_{ji}$.

One can easily verify $[e_{ij}, e_{kl}] \neq 0$ only in the following cases:

$$\begin{aligned} [e_{ij}, e_{jk}] &= e_{ik} && \text{if} && i \neq k \\ [e_{ij}, e_{ki}] &= -e_{kj} && \text{if} && j \neq k \\ [e_{ij}, e_{ji}] &= e_{ii} - e_{jj} && \text{if} && i \neq j. \end{aligned}$$

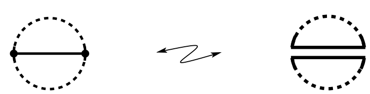
Hence, Lie bracket is given by tensor in $\mathfrak{gl}_N^* \otimes \mathfrak{gl}_N^* \otimes \mathfrak{gl}_N$:

$$[\cdot\,,\cdot] = \sum_{i,j,k=1}^{N} (e_{ij}^* \otimes e_{jk}^* \otimes e_{ik} - e_{ij}^* \otimes e_{ki}^* \otimes e_{kj}).$$

Transferring to $\mathfrak{gl}_N \otimes \mathfrak{gl}_N \otimes \mathfrak{gl}_N$ by above duality, tensor becomes

$$J = \sum_{i,j,k=1}^{N} (e_{ji} \otimes e_{kj} \otimes e_{ik} - e_{ji} \otimes e_{ik} \otimes e_{kj}).$$

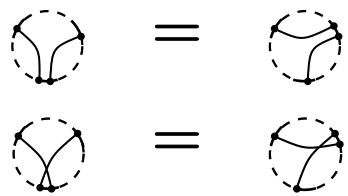
1.6.1 Theorem (D. Bar-Natan's computation of w.s. $\varphi^{St}_{\mathfrak{gl}_N}$). *Let $s(D)$ be number of connected components of the curve obtained by doubling all chords of a chord diagram D; i.e.,*

 .

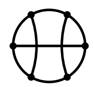
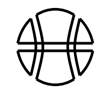
Then $\varphi^{St}_{\mathfrak{gl}_N}(D) = N^{s(D)}$.

Proof (hint). By matrices (e_{ij}) as chosen basis of $\mathfrak{gl}_N$, consider curve γ obtained by doubling the chords with ends labeled e_{ij} and e_{ji} in Wilson loop.

1.6.1 Remark. By definition, $s(D) = c-1$, where $c =$ number of boundary components of surface satisfying the 2-term relations:



.

1.6.1 Derivation. For $D =$ , we have the picture . Here $s(D) = 2$, hence $\varphi^{St}_{\mathfrak{gl}_N}(D) = N^2$.

1.6.1 Proposition. *The weight system $\varphi^{St}_{\mathfrak{gl}_N}(D)$ depends only on the intersection graph of D.*

Proof. $\varphi^{St}_{\mathfrak{gl}_N}(D)$ is defined by $s(D) = c-1$, for c boundary components (intersections), therefore function of genus of D, where genus depends only (proved!) on the intersection graph.

References

[1] M. Kontsevick. *Vassiliev's knot invariants*. I. M. Gelfand Seminar, Adv. Soviet Math. 16, Part 2, 137-150, Amer. Math. Soc., Providence, RI, 1993.

[2] R. Penrose. *Applications of negative dimensional tensors*. Combinatorial Mathematics and Its Applications, Proceedings of the Conference, Oxford, 1969, 221-244, Academic Press, London/Orlando, 1971.