Partition Function for Bipartite Graph Embedding

Matthew Bernard

mattb@berkeley.edu

Supported by the Lynn Bit Foundation, California

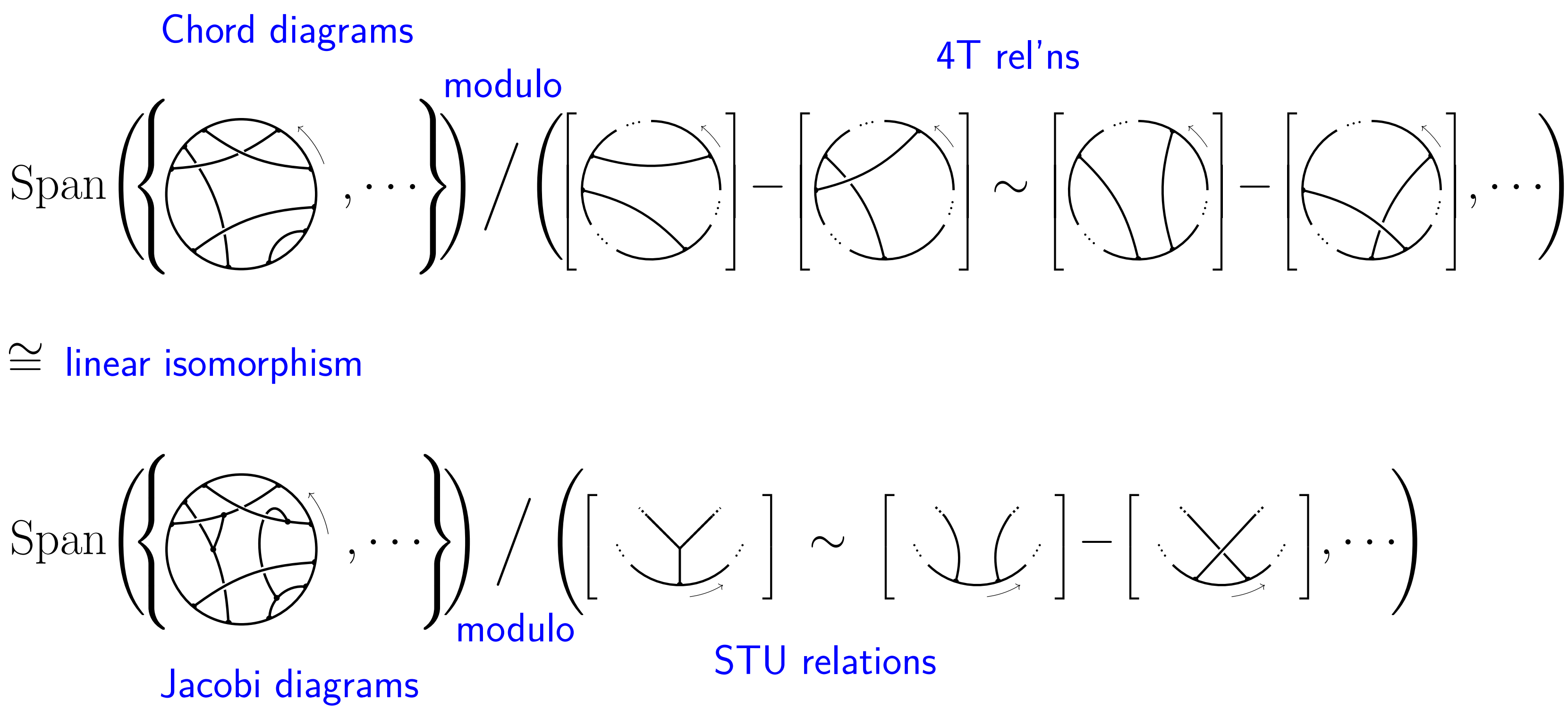
Abstract

We give partition function for oriented, embedded bipartite graph of specific genus, for observable. This generalizes notion of enumeration on Jacobi diagrams, more generally, classes on bipartite chord diagrams of even and odd number of chord intersections, by Pfaffian and Hafnian of constraint matrix that characterizes class of diagrams. We give some general statements, and apply obtained results in enumeration.

Keywords: Partition-function, observable, higher-genus

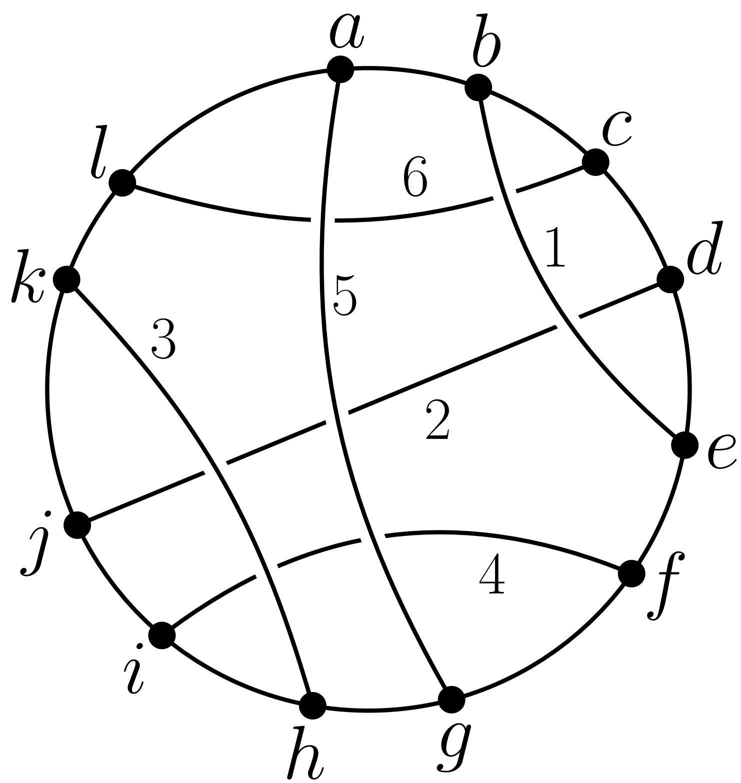
1 Characterizations

Jacobi diagram $\longleftrightarrow$ chord diagram (S.K. Lando, et. al.); e.g.:

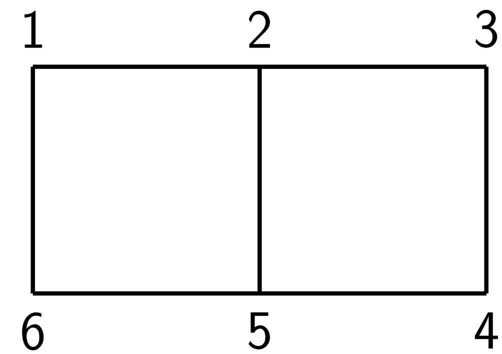


i.e. the equivalence of linear span of round chord diagrams modulo round 4T-relations to that of Jacobi diagrams modulo STU-relations.

Thus, given embedding, chord diagram $\longleftrightarrow$ intersection diagram



$\Longleftrightarrow$



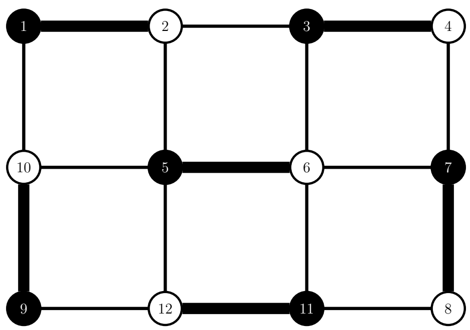
then UST problem: existence-uniqueness of conformal invariance of scaling limit in $\varepsilon \longrightarrow 0$ for classes of intersection graphs; rather efficient in bipartite perfect matching (NP-hard v. -complete). Where: UST (uniform spanning tree) on finite graph X is uniform measure on set of all spanning trees of X.

1.1 Basic definitions and properties

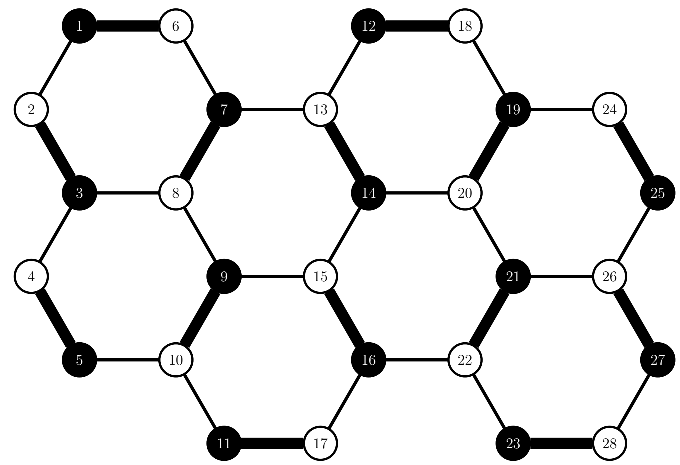
Bipartite $X \equiv \partial D = X^{\bullet} \sqcup X^{\circ}$ implies no odd cycle: no adjacent-blacks, -whites;

$$X^{\bullet} = \Big(\bullet : \ D = \bigsqcup_{k} \{ k^{\bullet}_{\sigma_\xi}, k^{\circ}_{\sigma_\eta} \}, \ 1 = |k^{*}_{\sigma_\xi} \subset D|, \ \emptyset = \bigcup_{k \neq \ell} \{ (k^{*}_{\sigma_\xi} \subset D), (\ell^{*}_{\sigma_\xi} \subset D) \} \Big).$$

Instance.

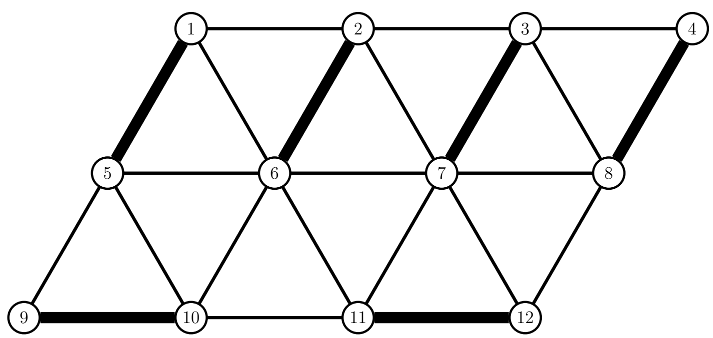


square grid domains.



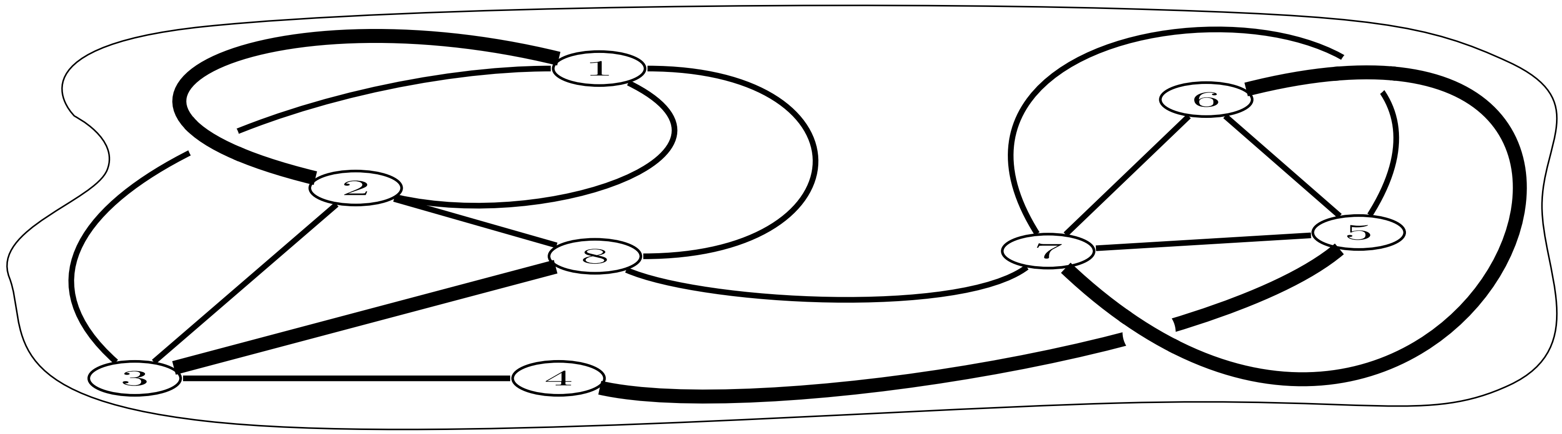
reg. hexagonal grid domains.

Non-instance.



$$\left(\begin{array}{c} \textit{no bipartite structure} \\ \textit{on triangular grids or odd lattices} \end{array} \right).$$

An embedding $X \subset \overline{\mathcal{M}}_g$, $g \gg$, $\mathsf{Aut}(\mathscr{D})$ *partition* is *equivalence class* $[\sigma]$ iff orientable compact $\overline{\mathcal{M}}_g$ closed X set $\mathscr{D}$ of perfect matchings D.



That is, $\forall\, k = (k_{\sigma_\xi} \equiv \sigma_\xi, k_{\sigma_\eta} \equiv \sigma_\eta)$: $\mathbb{1}_{k|D}\Big|_{=1 \iff D \supseteq \{k_{\sigma_\xi}, k_{\sigma_\eta}\},\ \text{else } 0}$ then

$$\sum_k \mathbb{1}_{k|D} = |\mathsf{Aut}(\mathscr{D})| / \Big((\frac{n}{2} - 1)!\, 2^{\frac{n}{2}}\, |\{\widetilde{\sigma}\}| \Big); \quad |\mathscr{D}| = \sum_{\sigma = [\sigma] = \widetilde{\sigma}} \prod_{\xi} \sum_k \mathbb{1}_{k|\{\sigma_{2\xi-1}, \sigma_{2\xi}\}}$$

where $X \subset \overline{\mathcal{M}}_g$ is CW cell-complex i.e. face $\mathscr{F} \approx$ topological disk: no hole;

$$\widetilde{\sigma} = \sigma\colon (\sigma_{2\xi-1} < \sigma_{2\xi};\ \sigma_{2\xi-1} < \sigma_{2\xi+1}); \quad \{[\sigma]\} \cong (\mathcal{S}_{\frac{n}{2}} \times \mathcal{S}_2^{\frac{n}{2}})^{(\mathsf{Aut}(\mathscr{D})/\mathsf{Aut}(D)) \cong [\sigma]}$$

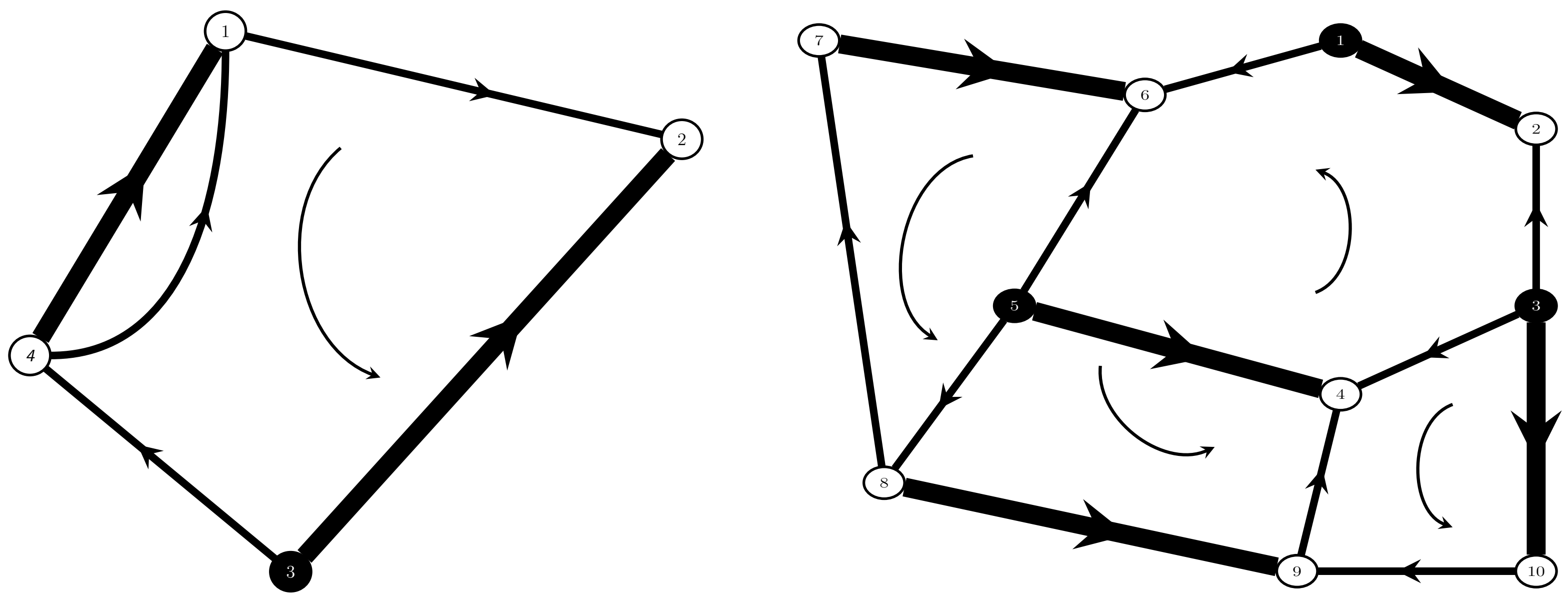
$$\sigma = (\sigma_1 \cdots \sigma_n) = ((\sigma_1\sigma_2) \cdots (\sigma_{n-1}\sigma_n)) \in \mathsf{Aut}(D) \cong \mathcal{S}_{\frac{n}{2}} \times \mathcal{S}_2^{\frac{n}{2}}$$

for strings $\big|\{\overline{[\sigma]}\}\big| = a\Gamma(\frac{a}{b})\, b^{(\frac{a}{b} - 1)}$; $a = |\mathsf{Aut}(\mathscr{D})|$, $b = \Gamma\big(\frac{n}{2} + 1\big)\Gamma^{n/2}(3)$.

Definition 1.1. *The* 1*-skeleton compact oriented CW complex* $\overline{\mathcal{M}}_g \supset D$ *is* $\mathscr{K}$ *digraph* $X = X^{\mathscr{K}}$ *if for all (counterclockwise)* k_ξ*-to-*k_η *i.e.* $\overrightarrow{k_{\xi\eta}}$ *of boundary orientation* $\overrightarrow{\partial\mathscr{F}} = \overrightarrow{\partial X} = \overrightarrow{k_{\xi\eta}}$ *for fixed* $\overrightarrow{k}$ *and sign* $\varepsilon^{\mathscr{K}}_{k_{\xi\eta}}$ *then*

$$\prod_{\overrightarrow{k_{\xi\eta}} = \overrightarrow{\partial\mathscr{F}}} \varepsilon^{\mathscr{K}}_{k_{\xi\eta}} = -1, \quad \forall\, \mathscr{F} \,\Bigg|\, -\varepsilon^{\mathscr{K}}_{k_{\eta\xi}} = \varepsilon^{\mathscr{K}}_{k_{\xi\eta}} = \begin{cases} -1 & \iff \overrightarrow{k} \text{ is } \overrightarrow{k_{\eta\xi}} \\ +1 & \iff \overrightarrow{k} \text{ is } \overrightarrow{k_{\xi\eta}} \end{cases}$$

i.e. parity $(|\mathscr{E}_{\mathscr{F}}|, \rho^+_{\mathscr{F}})$: $\rho^+_{\mathscr{F}} = |\mathscr{E}_{\mathscr{F}}| - \rho^-_{\mathscr{F}}$; $\rho^-_{\mathscr{F}} = \sum_{k \cap \partial\mathscr{F}} \mathbb{1}_{\varepsilon^{\mathscr{K}}_{k_{\xi\eta}} = -1}$; *for every* $\mathscr{F}$.



Definition 1.2. $X^{\mathscr{K}}_{\xi\eta}$, $\forall\, k$ *connecting* $k_\xi \equiv \xi$ *and* $k_\eta \equiv \eta$, *is well-defined:*

$$X^{\mathscr{K}}_{\xi\eta} = \sum_{1 = \mathbb{1}_{k|\{\xi,\eta\}}} \varepsilon^{\mathscr{K}}_{k_{\xi\eta}} \omega_k \;=\; -X^{\mathscr{K}}_{\eta\xi} \quad \Big|\, X^{\mathscr{K}}_{\xi\xi} = 0.$$

Derivation 1.1. If $\varepsilon^{\mathscr{K}}_{k_{\eta\xi}} := \varepsilon^{\mathscr{K}}_{k_{\xi\eta}} = +1$, then $(X^{\mathscr{K}}_{\xi\eta})$ is called adjacency matrix (resp. weighted adjacency matrix) for all $\omega_k = 1$ (resp. $\omega_k > 0$).

Derivation 1.2. For bipartite multiedge $X^{\mathscr{K}} \subset \overline{\mathcal{M}}_g$, then

$$X^{\mathscr{K}}_{\xi\eta} = -X^{\mathscr{K}}_{\eta\xi} = \begin{cases} \sum_{1 = \mathbb{1}_{k|\{\xi,\eta\}}} \omega_k & \text{if } \ \xi \bullet\!\!\Longrightarrow\!\!\circ\, \eta \ \text{ or } \ \xi \bullet\!\!\longrightarrow\!\!\circ\, \eta \\ -\sum_{1 = \mathbb{1}_{k|\{\xi,\eta\}}} \omega_k & \text{if } \ \xi \circ\!\!\Longleftarrow\!\!\bullet\, \eta \ \text{ or } \ \xi \circ\!\!\longleftarrow\!\!\bullet\, \eta \\ 0 \ \text{ if } \ \xi = \eta \ \text{ or } \ k \neq \ell, \ \ \forall\, k_{\xi\eta},\, \ell_{\xi\eta},\, k_{\eta\xi},\, \ell_{\eta\xi} \end{cases}$$

or

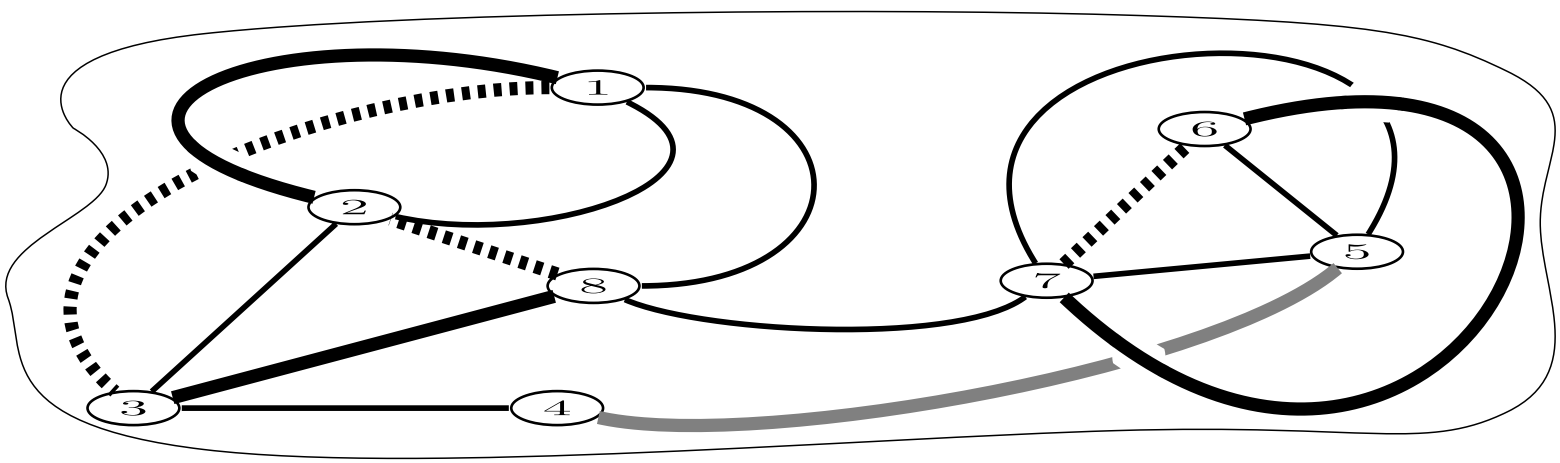
$$X^{\mathscr{K}}_{\xi\eta} = -X^{\mathscr{K}}_{\eta\xi} = \begin{cases} \sum_{1 = \mathbb{1}_{k|\{\xi,\eta\}}} \omega_k & \text{if } \ \xi \circ\!\!\Longrightarrow\!\!\bullet\, \eta \ \text{ or } \ \xi \circ\!\!\longrightarrow\!\!\bullet\, \eta \\ -\sum_{1 = \mathbb{1}_{k|\{\xi,\eta\}}} \omega_k & \text{if } \ \xi \bullet\!\!\Longleftarrow\!\!\circ\, \eta \ \text{ or } \ \xi \bullet\!\!\longleftarrow\!\!\circ\, \eta \\ 0 \ \text{ if } \ \xi = \eta \ \text{ or } \ k \neq \ell, \ \ \forall\, k_{\xi\eta},\, \ell_{\xi\eta},\, k_{\eta\xi},\, \ell_{\eta\xi}. \end{cases}$$

Remark 1.1. Bipartite, 2(odd) or hexagonal, $\mathscr{K}$ digraph is well-defined.

Derivation 1.3. Symmetric difference $D_\sigma \Delta D_\tau = D_\sigma \cup D_\tau \setminus D_\sigma \cap D_\tau$: 1-cycle homology $\mathcal{H}^1(X^{\mathcal{K}}; \mathbb{Z}_2) = \mathcal{H}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2)$ class of 1-chain complex $\mathcal{C}^1(X^{\mathcal{K}}; \mathbb{Z}_2)$; transition subgraph; finite, even-lengths n_α simple closed $\eta = \sum_\alpha \mathbb{1}_{C_\alpha | D_\sigma \Delta D_\tau}$ paths, traversing $(\xi_{n_{\alpha-1}+1}, (\xi_{n_{\alpha-1}+1}, \xi_{n_{\alpha-1}+2}), \ldots, \xi_{n_\alpha}, (\xi_{n_\alpha}, \xi_{n_{\alpha-1}+1}))$, in:

cycles $C_\alpha = (\xi_{n_{\alpha-1}+1}, \ldots, \xi_{n_\alpha})$; $\alpha = 1, \ldots, \eta$; $n_0 = 0$; such that, $\forall\, \alpha$:

$$\begin{array}{c} ((\xi_{n_{\alpha-1}+1}, \xi_{n_{\alpha-1}+2}), \ldots, (\xi_{n_\alpha-3}, \xi_{n_\alpha-2}), (\xi_{n_\alpha-1}, \xi_{n_\alpha})) \subseteq D_\sigma \\ ((\xi_{n_{\alpha-1}+2}, \xi_{n_{\alpha-1}+3}), \ldots, (\xi_{n_\alpha-2}, \xi_{n_\alpha-1}), (\xi_{n_{\alpha-1}+1}, \xi_{n_\alpha})) \subseteq D_\tau \end{array}.$$



Remark 1.2. D_σ, D_τ, are equivalent if $|D_\sigma \Delta D_\tau| = 0 \in \mathcal{H}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$; $D_\sigma, D_\tau = 1$-chain in cell-complex $\mathcal{C}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2)$; $\partial D_\sigma, \partial D_\tau = \mathcal{C}^0(\overline{\mathcal{M}}_g; \mathbb{Z}_2)$.

Lemma 1.1. *For all genus $g \gg$, the monomial sign*

$$\varepsilon^{\mathscr{K}}_{D} = (-1)^{t(\sigma)} \prod_{1=\mathbb{1}_{k|D}} \varepsilon^{\mathscr{K}}_{k_{\sigma_{2\xi-1}\sigma_{2\xi}}} \;\middle|\; \begin{array}{l} t(\sigma) := \sigma \longrightarrow (1 \cdots n) \\ \sigma \in \mathit{Aut}(D) \cong \mathcal{S}_{\frac{n}{2}} \times \mathcal{S}_{2}^{\frac{n}{2}} \end{array}$$

is invariant of Aut$(\mathscr{D})$.

Proof. $\varepsilon^{\mathscr{K}}_{D}$ is Aut(D) invariant by transposition of $\sigma_{2\xi-1}\sigma_{2\xi}$, with $(-1)^{t(\sigma)}$. Then let $D_1 \in \mathscr{D}$, $D_2 \in \mathscr{D}$, for $[\sigma]$, resp. $[\tau]$. With (even!) transition cycles C_α exactly odd $\rho^{-}_{C_\alpha} = \mathbb{1}_{C_\alpha|\xi^-}$ and $\rho^{+}_{C_\alpha} = \mathbb{1}_{C_\alpha|\xi^+}$, mononial i.e. composition $\gamma = \sigma \circ \tau$, and perhaps, $\sigma_{2\nu-1}\sigma_{2\nu} = \tau_{2\nu-1}\tau_{2\nu}$ for some ν, then

$$+1 \;=\; \varepsilon^{\mathscr{K}}_{D_1}\,\varepsilon^{\mathscr{K}}_{D_2} = \prod_{\alpha}\;\prod_{\xi\in C_\alpha}\;\prod_{\eta\in C_\alpha} \varepsilon^{\mathscr{K}}_{\sigma_{2\xi-1}\sigma_{2\xi}}\,\varepsilon^{\mathscr{K}}_{\tau_{2\eta-1}\tau_{2\eta}} \prod_{\nu} \left(\varepsilon^{\mathscr{K}}_{\sigma_{2\nu-1}\sigma_{2\nu}} = \varepsilon^{\mathscr{K}}_{\tau_{2\nu-1}\tau_{2\nu}}\right)^2$$

$$= \prod_{\alpha}\;\prod_{\xi\vee\xi^*\in C_\alpha}\;\prod_{\eta\vee\eta^*\in C_\alpha} \varepsilon^{\mathscr{K}}_{\sigma_{2(\xi\vee\xi^*)-1}\,\sigma_{2(\xi\vee\xi^*)}}\,\varepsilon^{\mathscr{K}}_{\tau_{2(\eta\vee\eta^*)-1}\,\tau_{2(\eta\vee\eta^*)}}$$

implies $\varepsilon^{\mathscr{K}}_{D_1} = \varepsilon^{\mathscr{K}}_{D_2}$ for $\mathbb{1}_{C_\alpha|\left(\xi^{\swarrow}\vee\xi^{*\swarrow}\vee\eta^{\swarrow}\vee\eta^{*\swarrow}\right)} = 1 \pmod 2$, $\forall\,\alpha$, by $\xi^*\vee\eta^*$ i.e. $\varepsilon^{\mathscr{K}}_{D_1} = \varepsilon^{\mathscr{K}}_{D_2}$, $\forall$ face $\rho^{-}_{\mathscr{F}}, \rho^{+}_{\mathscr{F}}$, and Aut$(D_1)$, Aut$(D_2)$ invariance in $\mathscr{D}$. □

Derivation 1.4. Suppose (Boltzmann) weights ω_k dimer energy $D\cap k_{\sigma_{2\xi-1}\sigma_{2\xi}} \longmapsto \Xi_k \mathbb{1}_{k|D} \mid \Xi\colon \mathscr{E}_X \longrightarrow \mathbb{R}_+;\ k_{\sigma_{2\xi-1}\sigma_{2\xi}} \longmapsto \Xi_k = -\ln \omega_k^{\mathcal{K}T}$ for the strict-sense positive partition function:

$$\mathcal{Z} \stackrel{\text{def}}{=\!=} \sum_D \omega_D; \quad \omega_D = \prod_{\mathbb{1}_{k|D}=1} \omega_k = e^{-\frac{\Xi_D}{\mathcal{K}T}}; \quad \Xi_D = \sum_k \Xi_k \mathbb{1}_{k|D}.$$

By $\mathbb{E}[\mathbb{1}_{k_{\sigma_\xi<\sigma_\eta}|D}\mathbb{1}_{\ell_{\sigma_\beta<\sigma_\gamma}|D}]\big|_{k=\ell;\{\xi,\eta\}=\{\beta,\gamma\}} = \mathbb{E}[\mathbb{1}_{k_{\sigma_\xi<\sigma_\eta}|D}]\big|_{=0 \Longleftrightarrow D \not\cap k_{\sigma_\xi<\sigma_\eta}}$ then the local observable i.e. dimer-dimer *correlation* (conditional probability):

$$\begin{aligned}\big\langle \mathbb{1}_{1_{\sigma_1<\sigma_2}|D}\cdots\mathbb{1}_{m_{\sigma_{2m-1}<\sigma_{2m}}|D}\big\rangle &\stackrel{\text{def}}{=\!=} \mathbb{E}\big[\mathbb{1}_{1_{\sigma_1<\sigma_2}|D}\cdots\mathbb{1}_{m_{\sigma_{2m-1}<\sigma_{2m}}|D}\big] \\ &= \mathbb{P}\big(D\cap 1_{\sigma_1<\sigma_2},\dots,D\cap m_{\sigma_{2m-1}<\sigma_{2m}}\big)\end{aligned}$$

which equals

$$\sum_D \mathbb{1}_{1_{\sigma_1<\sigma_2}|D}\times\cdots\times\mathbb{1}_{m_{\sigma_{2m-1}<\sigma_{2m}}|D}\,\mathbb{P}(D) = \frac{\sum_D \left(\prod_{k=1}^m \mathbb{1}_{k_{\sigma_{2k-1}<\sigma_{2k}}|D}\right)\omega_D}{\sum_D \omega_D}$$

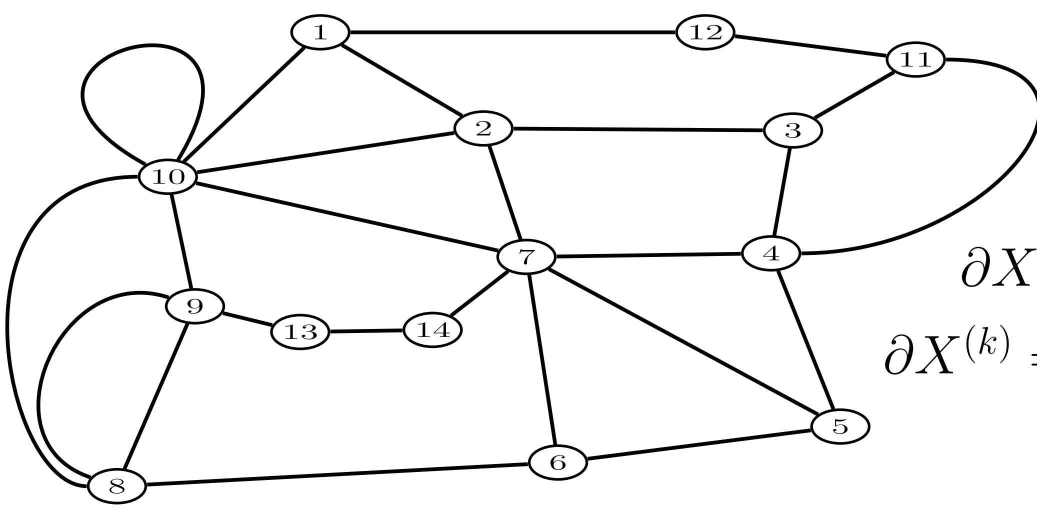
$$= \frac{1}{\mathcal{Z}}\mathcal{Z}_{[m:n]} = \frac{1}{\mathcal{Z}} \sum_{\substack{D:\\ 1=\prod_{k=1}^m \mathbb{1}_{k_{\sigma_{2k-1}<\sigma_{2k}}|D}}} \omega_D = \mathbb{P}(D) \Longleftrightarrow D = \bigcup_D \bigcap_k (D, k_{\sigma_\xi}, k_{\sigma_\eta}).$$

Proposition 1.1 (combinatorial correspondence). *Planar, spanning dual trees T^*, for all $\mathfrak{D} \longleftarrow$* ***Discrete surfaces****, implies*

$$\textit{\textbf{family (Dimers)}} \;\longleftrightarrow\; \textit{\textbf{family (Tilings)}}.$$

Proof. Generally for planar (i.e. non-intersecting) orientable $X^{\mathbb{R}^2} = X \subset \mathbb{R}^2$, the following applies:

(i) 2D cell complex $X \subset \mathbb{R}^2$:
0-cells, 1-cells, 2-cells $=$ vertices, edges, faces, resp.



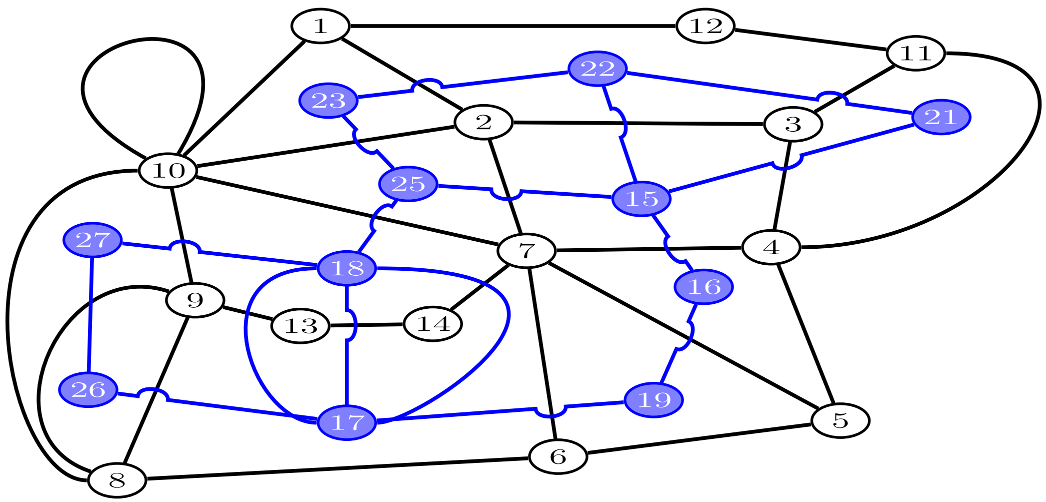
Disjoint interiors.

$$\partial X^{(k)} = ((k-1) \bmod 2)\text{-cell}$$

$$\partial X^{(k)} = \textit{boundary of two } k\textit{-cells},$$
$$k = 0, 1, 2.$$

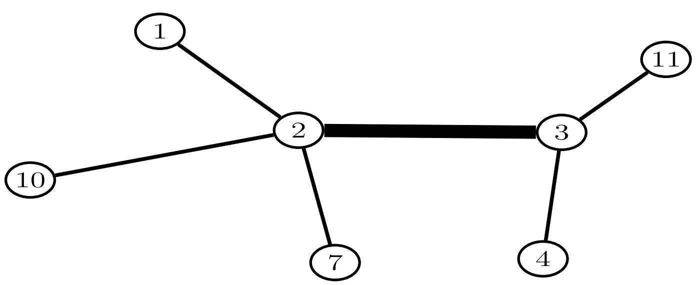
Remark 1.3. 1-skeleton CW complex $X^{\overline{\mathcal{M}}_g}$: orientable compact decompose.

(ii) 2D dual cell complex $X^* \subset \mathbb{R}^2$:
0-cells, 1-cells, 2-cells $=$ resp. "*centers*" of 2-cells, 1-cells, 0-cells of X.

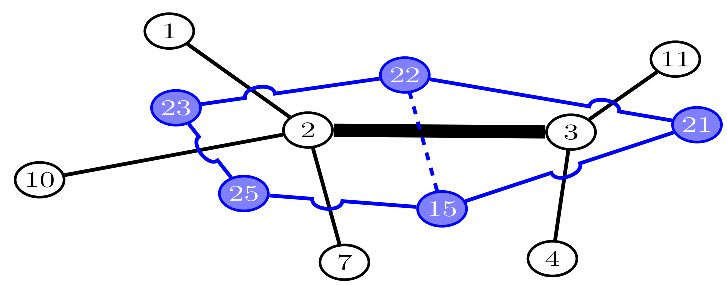


$X^* =$ dual cell complex to X.

(iii) For a dimer on X:



Unique pair of 2-cells on X^* share:

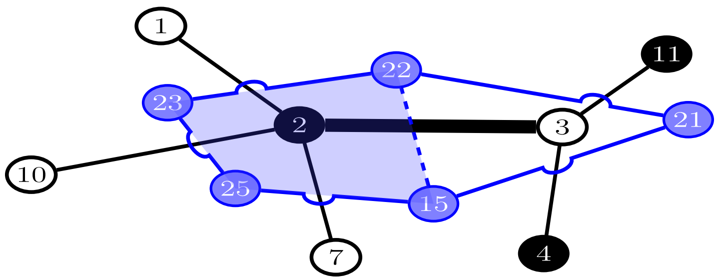


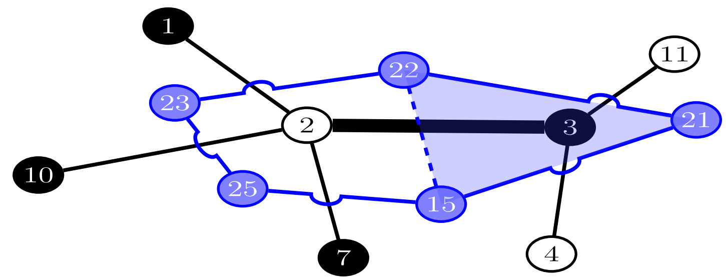
(iv) Therefore, the global bijection:

$$(\text{Dimers on } X) \quad \longleftrightarrow \quad \left(\begin{array}{l}\text{Tilings of } X^* \text{ by} \\ \textit{unique pair} \text{ of } 2\text{-cells}\end{array}\right).$$

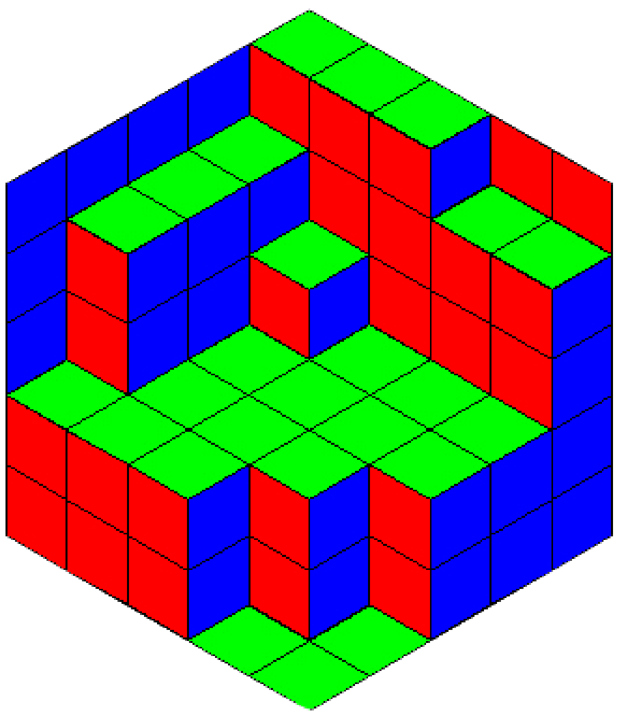
□

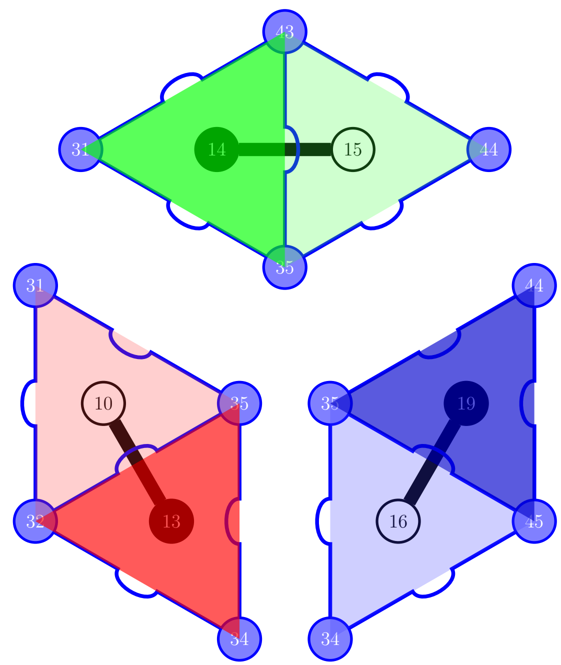
Remark 1.4. On bipartite graph, two-color tiles are admissible:

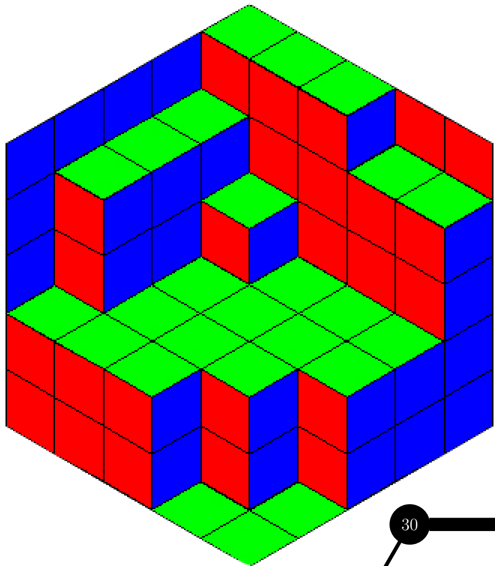




(Below: one-color tiles to the left, and two-color tiles to the right)

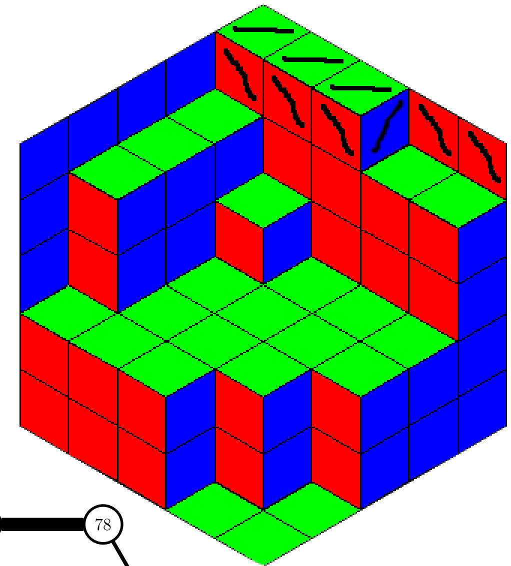


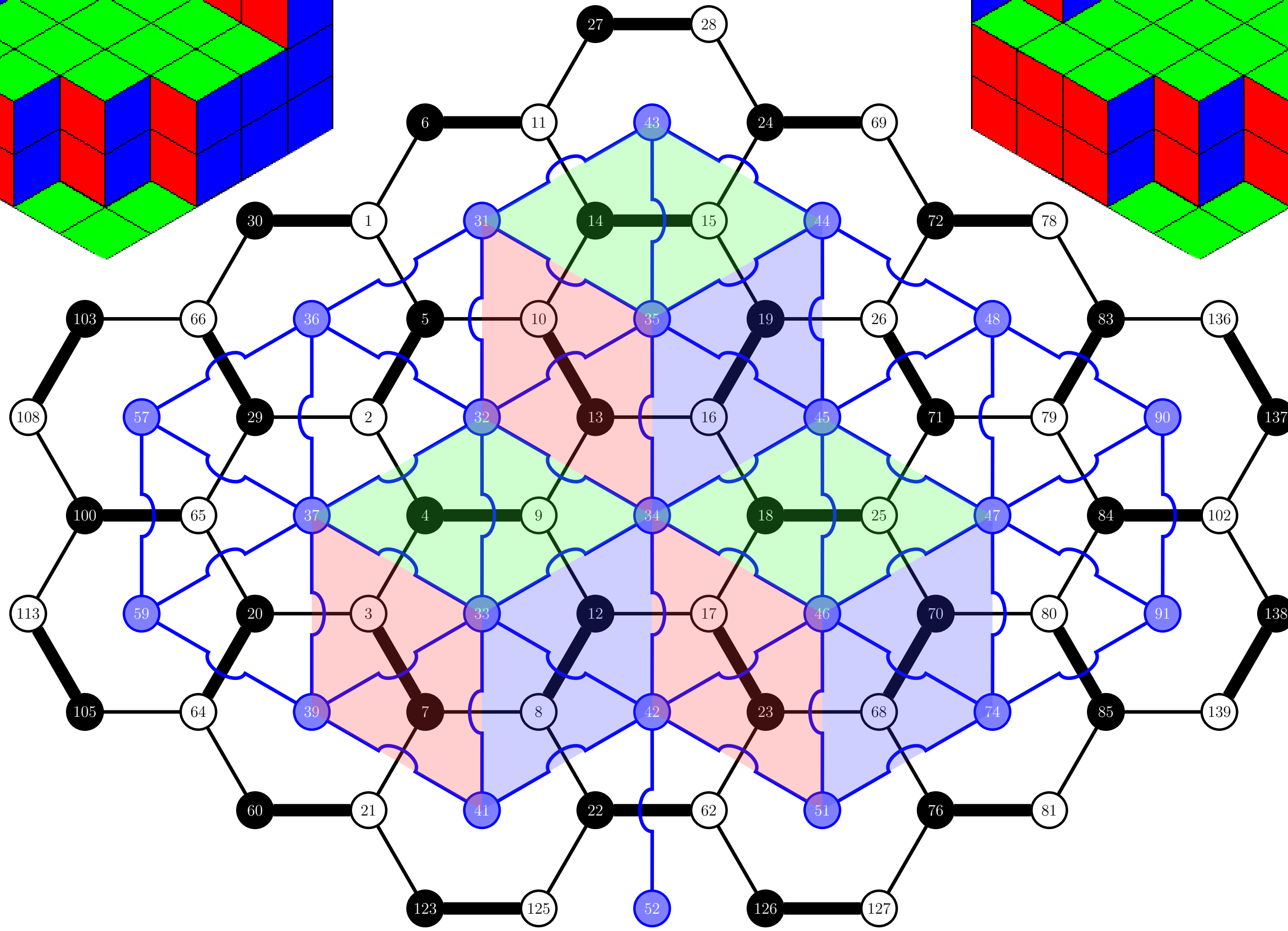




Cubes: 2D rhombus tiling

3D projection $\pi = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

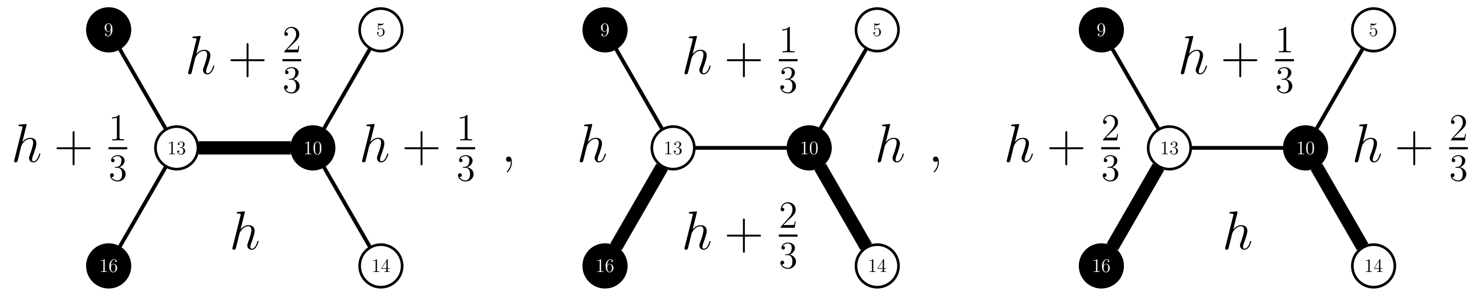




Definition 1.3. *Space $\mathscr{H}_X$ of height function $h_D, h, \forall X^*$, is whole of $\mathbb{Z}$:*

$$\boldsymbol{\mathscr{H}_X \overset{def}{=\!=} \{h_D \colon \mathscr{F}_X \longrightarrow \mathbb{Z}\}} \quad \Big| \quad \boldsymbol{\mathscr{D} \longleftarrow\!\!\!\!\dashv} \ \textbf{\textit{Bipartite surfaces}}.$$

Derivation 1.5. $\mathscr{H}_X$ is given on the bipartite hexagonal $X \subset \mathbb{R}^2$ by:



$$h(\mathscr{F}_{k_{\sigma_\xi}}) = \begin{cases} h(\mathscr{F}_{k_{\sigma_\xi - 1}}) + 1/3 & \text{if } k^{\bullet}_{\sigma_\xi} \text{ is left on crossing } k \not\cap D \\ h(\mathscr{F}_{k_{\sigma_\xi - 1}}) - 1/3 & \text{if } k^{\circ}_{\sigma_\xi} \text{ is left on crossing } k \not\cap D \end{cases}; \quad h(\mathscr{F}_{k_0}) = 0.$$

for any $D \in \mathscr{D}$ with base-face normalization $h_D(\mathscr{F}_{k_0}) = 0$.

Theorem 1.1. *$h_D = h$, i.e. independent of D. And, $h_{D_1 \Delta D_2} = h_{D_1} - h_{D_2}$.*

Proof. By directional flow $\widetilde{\omega}$, for divergence-free notion, consider curl sum

$$d_X = \sum_{\mathscr{F}} d_{\mathscr{F}} = \sum_{\mathscr{F}} \sum_{\mathbb{1}_{k|\mathscr{F}} = 1} \widetilde{\omega}_{k_{\{\sigma_\xi, \sigma_\eta\}}}.$$

Take

$$d^\star_{\sigma_{\xi_{D_1D_2}}} = d^\star_{\sigma_{\xi_{D_1}}} - d^\star_{\sigma_{\xi_{D_2}}} \;\Bigg|\; d^\star_{\sigma_{\xi_D}} = d^\star_{\sigma_\xi} = \sum_k \sum_{\xi\neq\eta} \left(\widetilde{\omega}_{k_{\{\sigma_\xi,\sigma_\eta\}}} = \begin{cases} +1 & \text{if } D\cap k^\bullet_{\sigma_\xi} \\ -1 & \text{if } D\cap k^\circ_{\sigma_\xi} \\ 0 & \text{otherwise} \end{cases} \right).$$

Then $d^\star_{\sigma_{\xi_{D_1D_2}}}$, resp. d_X, is zero iff $\mathscr{F}_X$ is all co-cycles, hence the claims. $\square$

Remark 1.5. X^* cubes π_{ab} skew plane partition i.e. diagonal slices sequence

$$\{\lambda\colon\ \lambda\supset\mu\}\ \mid\ \lambda(t) = \big(\pi_{a,\,a+t}\in\mathbb{N}\colon\ a\geqslant\max(0,-t),\ \forall\, t\in\mathbb{Z}\big)$$

generalizes to 3D array partition $\pi = (\pi_{ab}\colon (a,b)\in\mathbb{N}^2 \mid \pi_{ab}=0,\ \forall\, a+b\gg 0)$ for finite monotone array $(\pi_{ab}\geqslant\pi_{a+r,\,b+s},\ \forall\, r,s\geqslant 0)$.

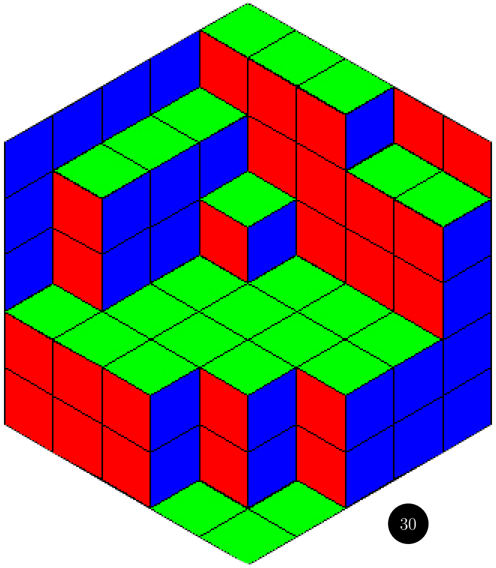
Remark 1.6. An array π is uniquely X^* bijection projection map

$$\mathbb{R}^3\longmapsto\mathbb{R}^2\supset\{(t,h)\}\colon\ t=y-x,\ \ h=z-(y+x)/2,\ \ \forall\,(x,y,z)\in\mathbb{R}^3$$

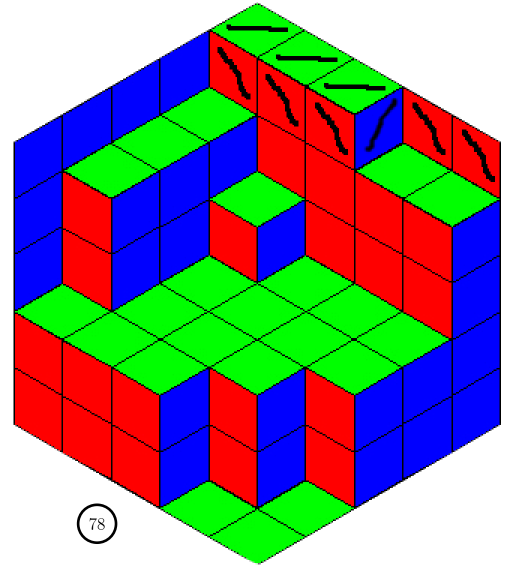
for all cubes mod $\mathbb{Z}^3_{\geqslant 0}$ projection, with boundary (base) condition $(0,0,0)$.

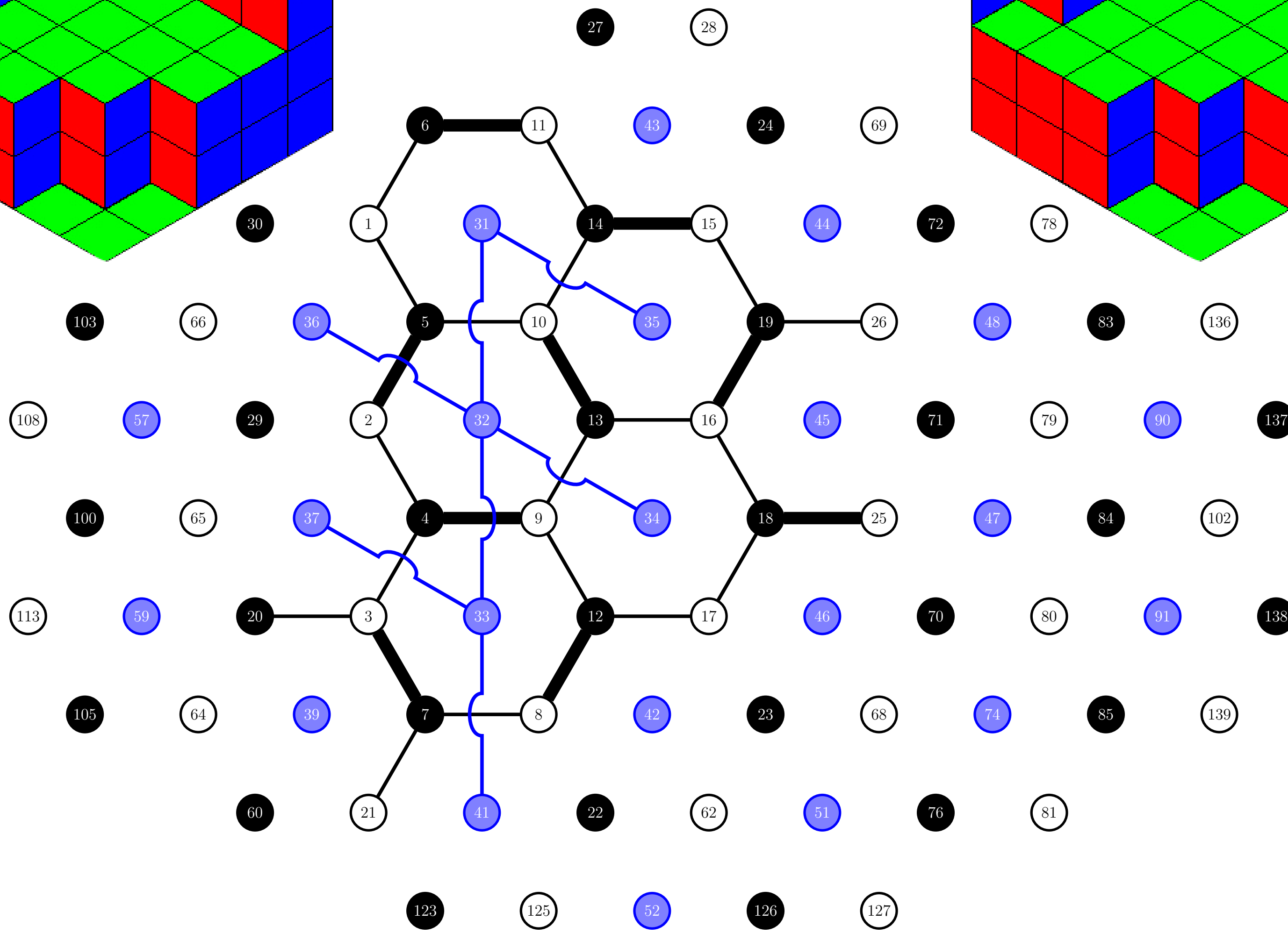
Centers of the horizontal hexagonal tiling are given by:

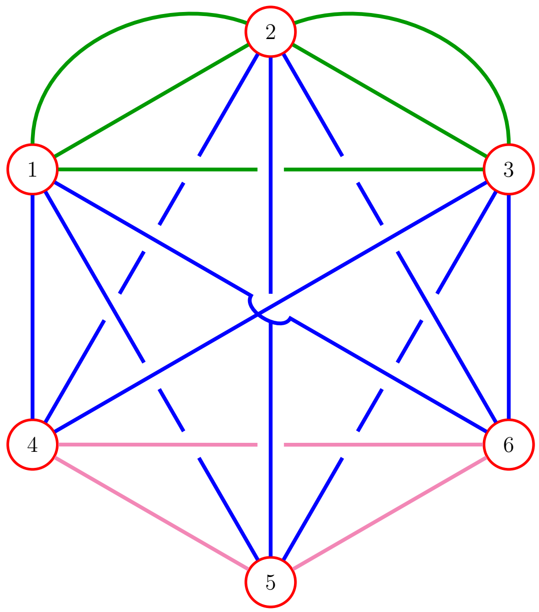
$$\pi_C=\Big\{\Big(a-b,\ \ \pi_{ab}-\frac{1}{2}(a+b-1)\Big)\Big\}\ \subset\ \mathbb{Z}\times\frac{1}{2}\,\mathbb{Z}.$$



Cubes: 2D mixing algorithm process







$$\begin{pmatrix} \begin{array}{ccccc|c|cccc} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{array} \end{pmatrix}$$

$\mathscr{D}$ (left)
$\mathscr{H}_X$ (right)

Proposition 1.2 (bijection).

$$\{\textit{Dimers on } X\} \underset{\textit{bijection}}{\cong} \{\textit{height functions}\}.$$

Proof. Follows from the combinatorial correspondence. □

Lemma 1.2.

$$\textit{Prob}(h) = \frac{1}{\mathcal{Z}} \prod_{\mathcal{F}} q_{\mathcal{F}}^{h_{\mathcal{F}}} \;\Big|\; \mathcal{Z} = \sum_{h \in \mathcal{H}_X} \prod_{\mathcal{F}} \Big(\prod_{k_{\xi\eta} \in \varepsilon_{\partial\mathcal{F}}} \omega_k^{\varepsilon^{\mathcal{K}}_{k_{\xi\eta}}} \Big)^{h_{\mathcal{F}}}$$

for the "fundamental" invariant parameter $q_{\mathcal{F}}$; $\forall\, \mathcal{F} \in \mathcal{F}_{X^{\mathcal{K}}} \subseteq X^{\mathcal{K}} \subset \overline{\mathcal{M}}_g$.

Proof. Follows from Proposition 1.2. □

Theorem 1.2. *For* $\mathcal{D}$ *and* $\mathcal{H}_X$ *of a genus embedding,*

$$\textit{Prob}(D) = \frac{1}{\mathcal{Z}} \prod_{\mathcal{F}} q_{\mathcal{F}}^{h_{\mathcal{F}|D}}, \qquad \mathcal{Z} = \sum_{D} \prod_{\mathcal{F}} \Big(\prod_{k_{\xi\eta} \in \varepsilon_{\partial\mathcal{F}}} \omega_k^{\varepsilon^{\mathcal{K}}_{k_{\xi\eta}}} \Big)^{h_{\mathcal{F}|D}}.$$

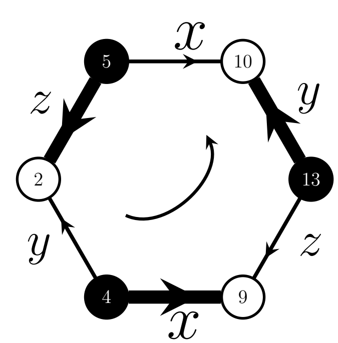
Proof. Follows by the bijection and the prior lemma. □

Remark 1.7. Prob(D) is "gauge" invariant measure: $\omega_\xi \longmapsto s(\xi_+)\, \omega_\xi\, s(\xi_-)$.

Case 1.1.

(i) Uniform distribution:

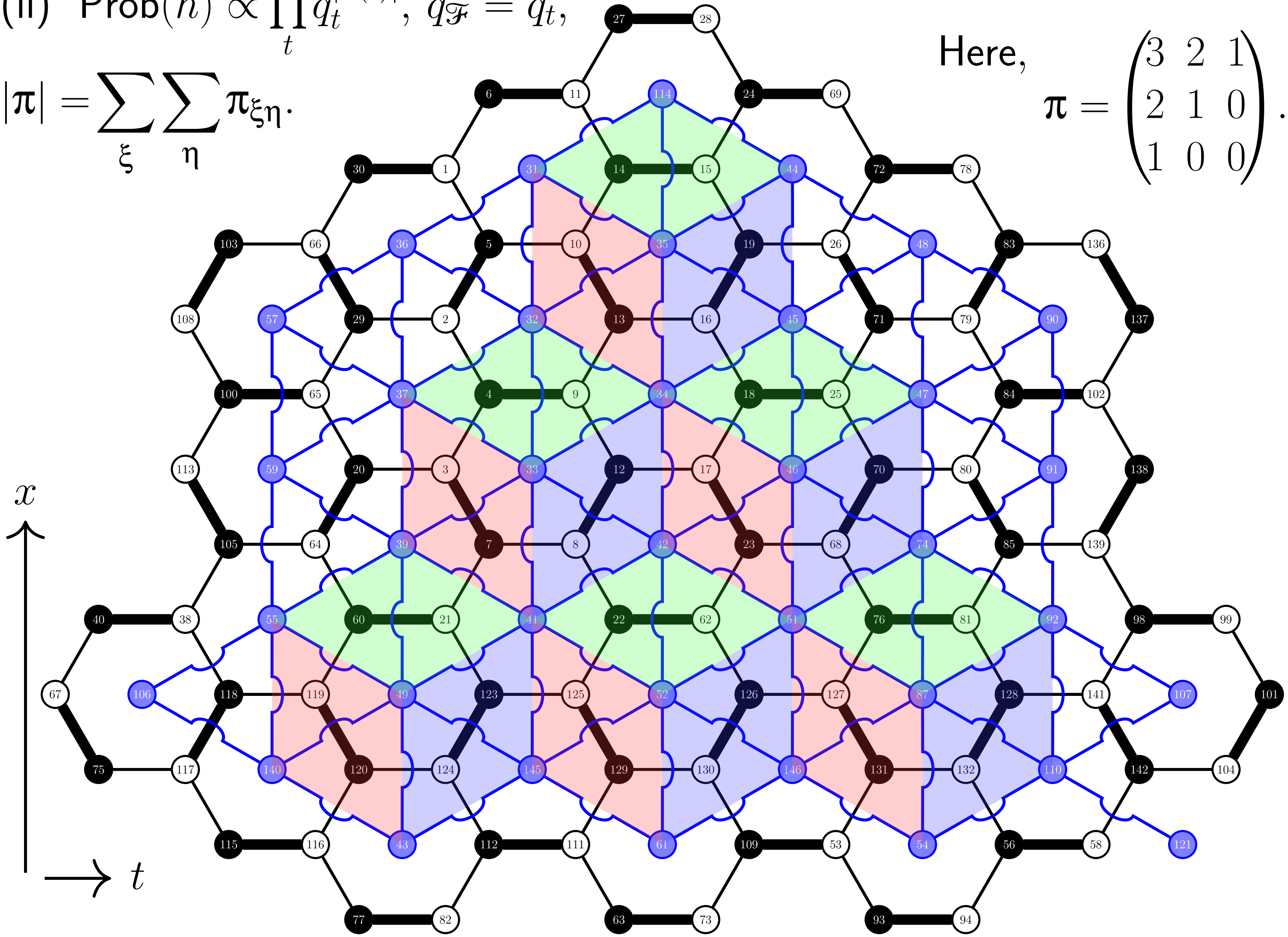
$$q_{\mathcal{F}} = 1 = x^{-1} y z^{-1} x y^{-1} z.$$



(ii) $\mathrm{Prob}(h) \propto \prod_{t} q_t^{|\pi(t)|}$, $q_{\mathscr{F}} = q_t$,

$$|\pi| = \sum_{\xi} \sum_{\eta} \pi_{\xi\eta}.$$

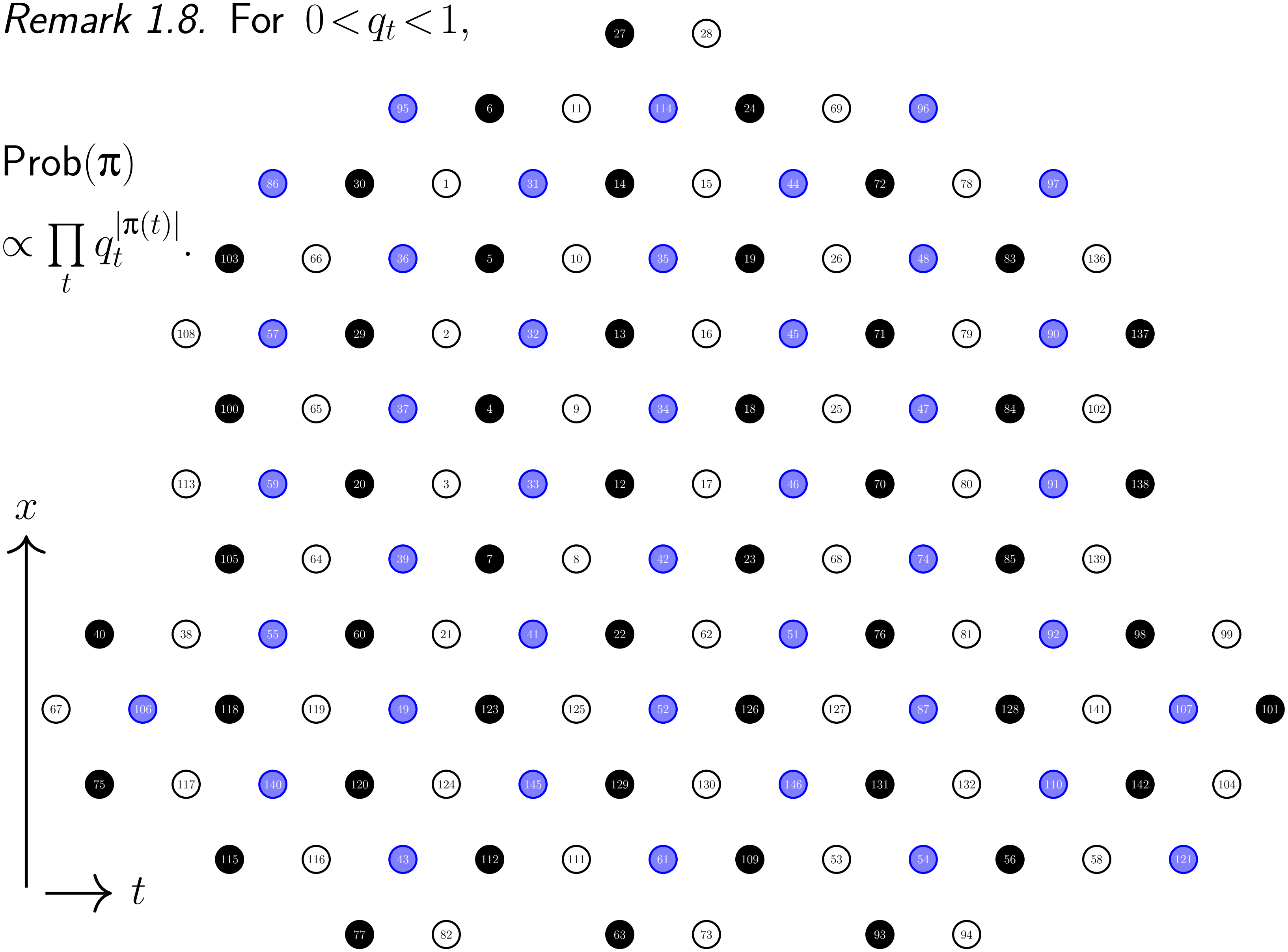
Here, $\pi = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$



Remark 1.8. For $0 < q_t < 1$,

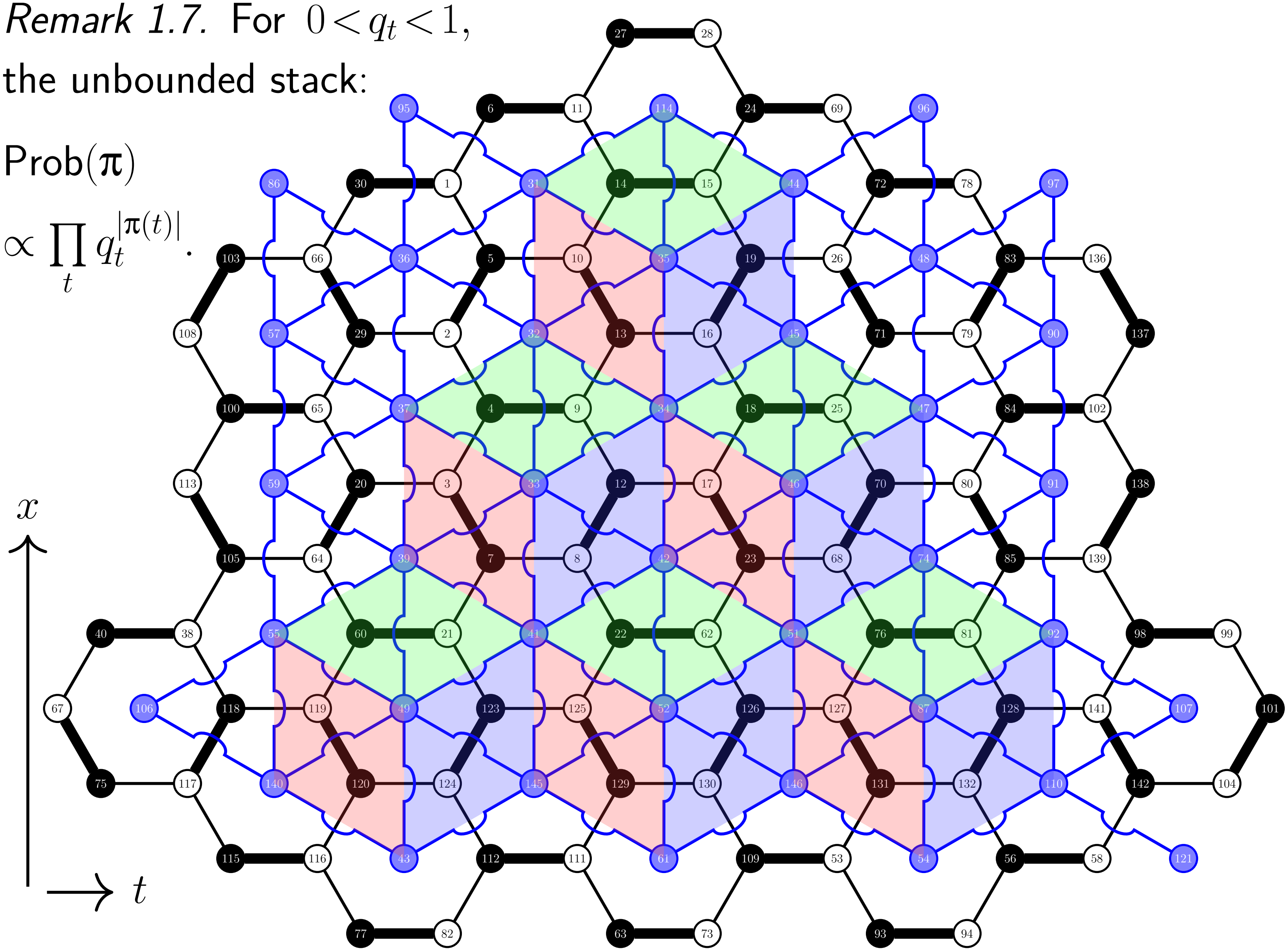
$\text{Prob}(\pi)$

$\propto \prod_t q_t^{|\pi(t)|}$.



Remark 1.7. For $0 < q_t < 1$, the unbounded stack:

$$\mathrm{Prob}(\boldsymbol{\pi}) \propto \prod_t q_t^{|\boldsymbol{\pi}(t)|}.$$



1.2 What is known

1.2.1 Order of (+) or (−), fixed $g \geqslant 0$, Pfaffians in $\mathcal{Z}$

Kasteleyn (1963). For $g = 0$, $\mathcal{Z} = \pm$ Pfaffian of $\mathscr{K}$ matrix $(X^{\mathscr{K}}_{\xi\eta})$.

Kasteleyn (1963). For $g = 1$, $\mathcal{Z} =$ linear in 4 Pfaffians; 3 "+", 1 "−".

Kasteleyn (1963). For $g \geqslant 2$, $\mathcal{Z} =$ conjecture: 2^{2g} Pfaffians, appearing mysteriously i.e. proof was not given, at least not published.

1.2.2 Combinatorial representation of (+), (−), in $\mathcal{Z}$

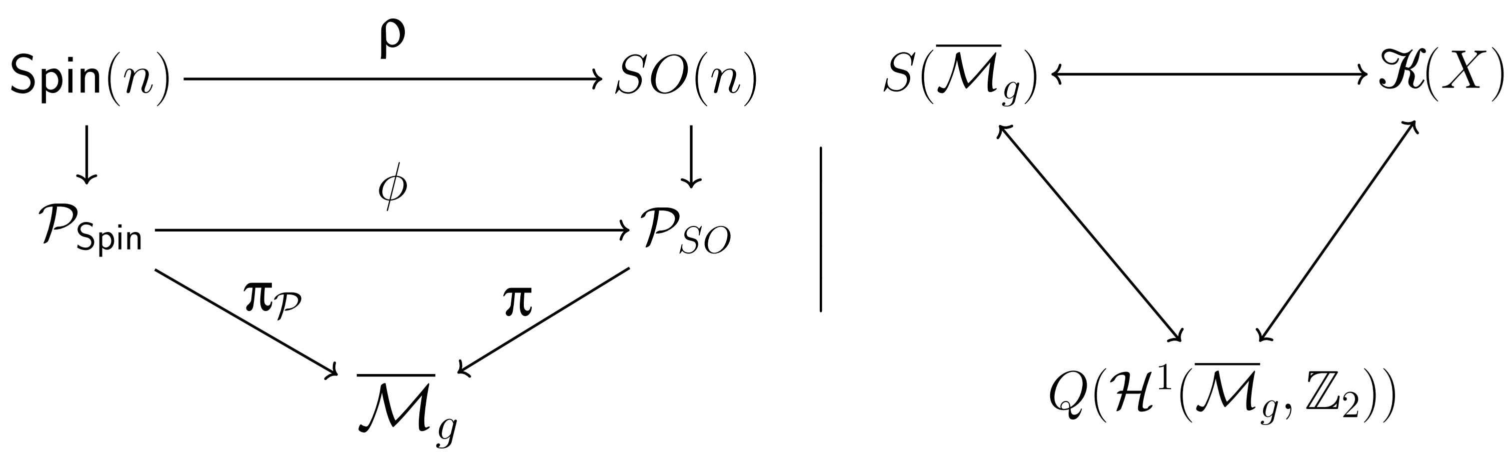
Gallucio & Loebl (1999). $\mathcal{Z} := \pm 1$; $\overline{\mathcal{M}}_g$ compact orientable.

Tesla (2000). $\mathcal{Z} := \sqrt{-1}$ and ± 1; $\overline{\mathcal{M}}_g$ non-orientable.

Cimazoni & R. (2004, 2005). $\mathcal{Z} := \pm 1$ by spin structure.

Cimasoni (2006). $\mathcal{Z} := \sqrt{-1}$ by pin-minus structure for double-cover; $\overline{\mathcal{M}}_g$ non-orientable; a Tesla (2000) topological model $\cong$ spin structure's ± 1.

Lie group G, transitive, free G-action, homogeneous G-space $\overline{\mathcal{M}}_g$ principal bundle lift $\mathcal{P}_{\mathsf{SO}(E)}$ to $\mathcal{P}_{\mathsf{Spin}(E)}$, spin structure $S = (\mathcal{P}_{\mathsf{Spin}}, \phi) \cong$ partition (Pfaffians); i.e. $|S(\overline{\mathcal{M}}_g)| = |\mathcal{K}(X)|$ via vector field $\mathbb{V}$ oriented vector bundle E cohomologous space $\mathcal{H}$ of even-index singularity (zeroes) holomorphic in zdz polynomial L quadratic form $\sqrt{L(z)}$. Commutatively:



$$\pi_{\mathcal{P}} = \pi \circ \phi;\ \ \phi(p,q) = \phi(p)\rho(q);\ \ p \in \mathcal{P}_{\mathsf{Spin}};\ \ q \in \mathsf{Spin}(n)$$

for $Q \cong \sqrt{L(z)}$; $\forall\, g \ngtr 2$, tangent bundle equivariant 2-fold covering bundle map ϕ, principal $\mathsf{Spin}(n)$ bundle $\pi_{\mathcal{P}}$; double covering map ρ of spin group $\mathsf{Spin}(n)$ to $\mathsf{SO}(n)$ double-cover; k-fold covering $\pi : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ as structure group $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ principal bundle, for standard circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

1.2.3 Asymptotics of bipartite observable (Pfaffians)

R. et al. (2005). For height functions $h \in \mathbb{Z}$, face-weights $q_{\mathcal{F}}$, $\forall\, g \geqslant 2$,

$$\mathcal{Z}(\text{bipartite}) = \text{Const.} \times \sum_{h} \prod_{\mathcal{F}} q_{\mathcal{F}}^{h(\mathcal{F})} \quad ; \quad \mathcal{Z} = \frac{1}{2^g} \sum_{\mathfrak{T} \in S(\overline{\mathcal{M}}_g)} \text{Arf}(q_{\mathfrak{T}}^{\mathcal{K}}) \cdot \text{Pf}(X_{\mathfrak{T}}^{\mathcal{K}})$$

and, as $|X| \longrightarrow \infty$, $q_{\mathcal{F}} \longrightarrow 1$, in Seiberg-Witten conjecture (Gaussian free theory) entropy, $\mathcal{Z}$ is scaling-limit path integral:

$$\mathcal{Z} = \int \exp\left\{ -\frac{1}{2} \left(\int_{\overline{\mathcal{M}}_g} (\partial \Phi)^2 \, d^2 x + \int_{\overline{\mathcal{M}}_g} \lambda(x)\, \Phi(x) \right) \right\}$$

where the term $q_{\mathcal{F}}^{h(\mathcal{F})}$ contributes to **R.H.S** linear multiple $\lambda(x)\,\Phi(x)$ by:

$$q_x = \xi^{-\varepsilon \cdot \lambda(x)} \quad \Big| \; \varepsilon = \text{lattice step; } \lambda = \text{logarithmic scale, as } \varepsilon \longrightarrow 0.$$

Moreover, in Alvarez-Gaumé, Moore, Nelson & Vafa (1986), studying Fermi and Bose partition correspondence on Riemann surfaces,

$$\textbf{R.H.S.} \sim \sum_{\mathfrak{T} \in S(\overline{\mathcal{M}}_g)} \text{Arf}(\mathfrak{T}) \times |\Theta(z \,|\, \mathfrak{T})|^2 \quad \Big| \; \omega \text{ determines } z.$$

Remark 1.9. Conjecture: In large thermodynamic scaling limit asymptotics, the observable decaying linearly goes to the critical-weight

$$e^{\textbf{Volume}} \times \textbf{ the free energy}$$

where the next leading term is sum of theta functions, and square of each theta function is the next leading asymptotics of each of the Pfaffians.

The conjecture was confirmed by:

(i) **Ferdinand (1967).** *On square-grid torus.*

(ii) **Costa-Santos & McCoy (2002).** *Numerically*:

$$\mathsf{Arf}(\mathfrak{T}) \times |\Theta(z\,|\,\mathfrak{T})|^2 \;\;\Big|\;\; g \geqslant 2.$$

That is, works but without proof; hence, remains a conjecture.

Remark 1.10. (i) $\mathcal{Z}$ is surface glueable (summable) on boundary spins.

(ii) "Higher" spin-structure is unknown, perhaps in para-polynomial theory.

(iii) Observable method is non-deterministic sophistication, unlike $\mathrm{d}\log\boldsymbol{\omega}$.

Plan

1. Operators

(i) Prove $\mathcal{Z}$ invariants for all genus g closed, bipartite T^*

(ii) Prove the $\mathcal{O}(\cdot)$ observable for all fixed sufficient-large genus $g \gg 0$

2. Vertex algebras

(i) Reformulate Kasteleyn Grassmann integral on transfer matrices Pf$(\cdot)$

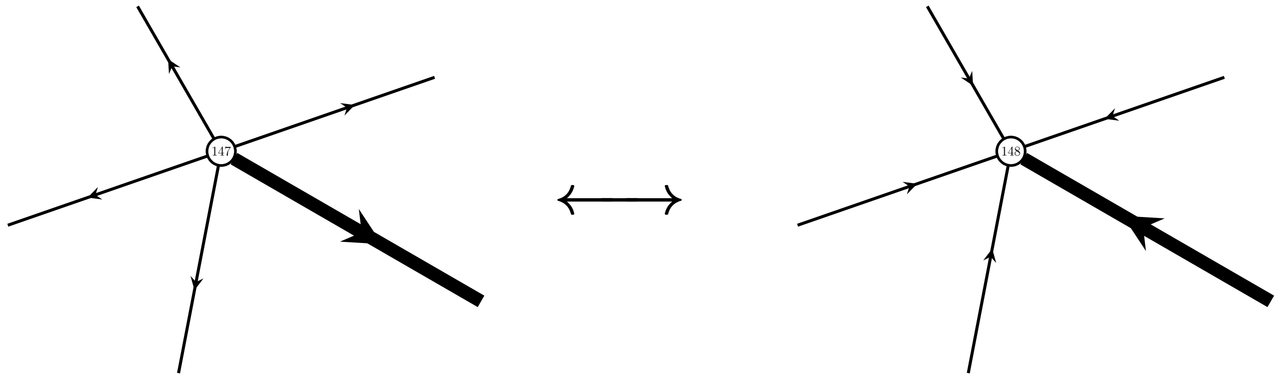
(ii) Prove Grassmann kernel convergence for T^* special genus g domain

(iii) Obtain $\mathbb{R}$ log scaling asymptotics, limit shape variational principle

(iv) State the conjecture for Green's function $\langle \cdot \rangle$ in large deviation

2 Operators

Definition 2.1. *Two orientations are equivalent iff invertible map holds:*



Theorem 2.1. *All orientations $\mathscr{K}$ for all $X^{\mathscr{K}} \subset \mathbb{R}^2$ are equivalent.*

Proof. Mark two $\mathscr{K}$ orientations $\mathscr{K}_-$, $\mathscr{K}_+$ on k end ξ, resp. η, $\forall \mathscr{F}$, then

$$\varepsilon_k^{\mathscr{K}_-} = \varepsilon_k^{\mathscr{K}_+} \cdot \sigma_k^{\mathscr{K}_- \mathscr{K}_+}, \quad \varepsilon_k^{\mathscr{K}_+} = \varepsilon_k^{\mathscr{K}_-} \cdot \sigma_k^{\mathscr{K}_- \mathscr{K}_+} \quad \Big| \, \sigma_k^{\mathscr{K}_- \mathscr{K}_+} = \varepsilon_k^{\mathscr{K}_-} \cdot \varepsilon_k^{\mathscr{K}_+}$$

i.e. $\mathscr{K}_- \longrightarrow \mathscr{K}_+$ (resp. $\mathscr{K}_+ \longrightarrow \mathscr{K}_-$) by $\sigma_k^{\mathscr{K}_- \mathscr{K}_+}$ multiplying $\mathscr{K}_-$ (resp. $\mathscr{K}_+$) at every vertex; and, $\mathscr{K}_- \longleftrightarrow \mathscr{K}_+ \longleftrightarrow$ equivalence class $[\mathscr{K}]$ in simple reversal of orientations around vertices by $-1 = \sigma_k^{\mathscr{K}_- \mathscr{K}_+} := \pm 1$. $\square$

Corollary 2.1. *Equivalence class $[\mathscr{K}]$ is unique for all $X^{\mathscr{K}} \subset \mathbb{R}^2$.*

Proof. $\exists$ one homotopy class of loops i.e. $\mathbb{R}^2$ trivial fundamental group. $\square$

Theorem 2.2. *Equivalence-classes of $\mathscr{K}$ for all $X \subset \overline{\mathcal{M}}_g$ is exactly 2^{2g}.*

Proof. The isomorphisms $\{[\mathscr{K}]\}$ are in characteristic-2 field κ affine closure $\mathsf{Sym}^2_\kappa(V^\wedge)$ of non-degenerate, skew-symmetric quadratic bilinear form

$$q(\alpha+\beta) = q(\alpha) + q(\beta) + \alpha \boldsymbol{\cdot} \beta \;\Big|\; q\colon V \otimes V \longrightarrow \kappa\,,\ \forall\, \alpha, \beta \in \mathcal{H}^1 = V \otimes V$$

in first homology space $\mathcal{H}^1 \ni \alpha$, for

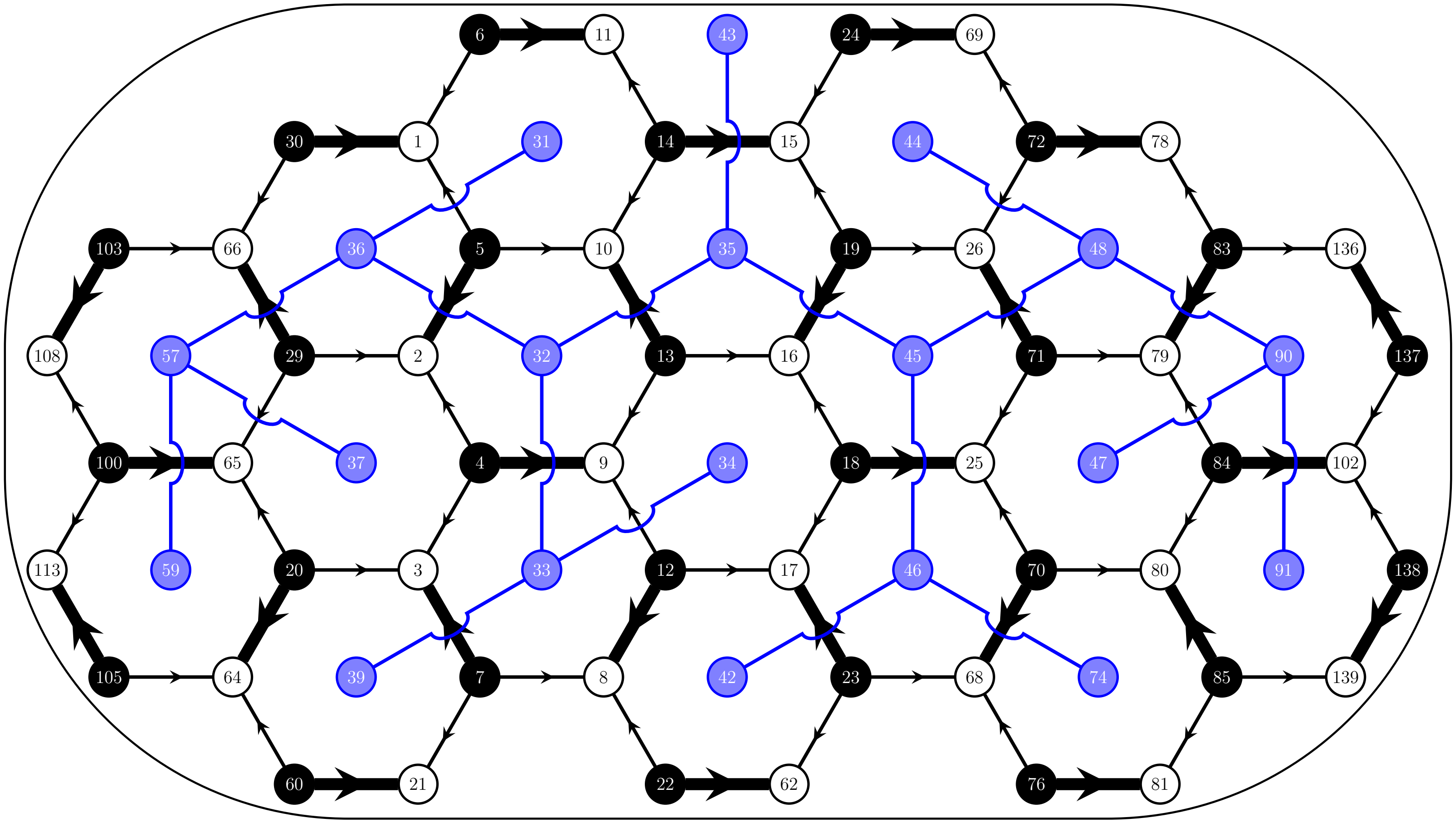
$$\frac{1}{\sqrt{|\mathcal{H}^1|}} \sum_{q \,\in\, (\mathcal{H}^1, \boldsymbol{\cdot})} (-1)^{\mathit{Arf}(q)+q(\alpha)} = 1 \;\Bigg|\; \mathit{Arf}(q) = \sum_{\{\xi,\eta\}} q(\xi)q(\eta) \;\in\; \kappa/f(\kappa) \subset \mathbb{Z}_2$$

where $\{\xi, \eta\}$ are symplectic basis pairs for symplectomorphisms $V \longrightarrow V$, Lang's isogeny $f\colon \kappa \longrightarrow \kappa \mid x \longmapsto x^2 - x \in \mathsf{Gal}/\mathbb{F}_2$ (2-element Galois field).

By continuity $\psi: X^{\mathscr{K}} \longrightarrow \overline{\mathcal{M}}_g$, every $\overline{\mathcal{M}}_g \backslash \psi(X^{\mathscr{K}})$ connected-components (ψ-faces $\mathscr{F}$) $\approx$ open disk, i.e. $\chi(X^{\mathscr{K}}) = \chi(\overline{\mathcal{M}}_g)$ in Euler-Poincaré bound $|\mathscr{V}_{X^{\mathscr{K}}}| - |\mathscr{E}_{X^{\mathscr{K}}}| + |\mathscr{F}_{X^{\mathscr{K}}}| = \chi(X^{\mathscr{K}}) \geqslant \chi(\overline{\mathcal{M}}_g)$. Vanishing composition $\partial_1 \circ \partial_2$ of boundary operators $\partial_2\colon \mathcal{C}_2 \longrightarrow \mathcal{C}_1$, $\partial_1\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_0$ for basis $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$ of 2D cell-complex $\mathscr{V}_{X^{\mathscr{K}}}, \mathscr{E}_{X^{\mathscr{K}}}, \mathscr{F}_{X^{\mathscr{K}}}$, resp. implies 1-cycle space superset $\mathsf{Ker}(\partial_1)$ of 1-boundary space $\partial_2(\mathcal{C}_2)$. Hence, independent of $X^{\mathscr{K}}$ but depending only on genus g: $|\mathcal{H}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2)| = |\mathcal{H}^1(X^{\mathscr{K}}; \mathbb{Z}_2)| = |\mathsf{Ker}(\partial_1)/\partial_2(\mathcal{C}_2)| = 2^{2g}$. $\square$

Theorem 2.3 (existence). $\exists\, \mathscr{K} \iff |\mathscr{V}_X| =$ *even,* $\forall\, X \subset \overline{\mathcal{M}}_g$ *bipartite.*

Proof. Following a rooted spanning dual tree T^*:



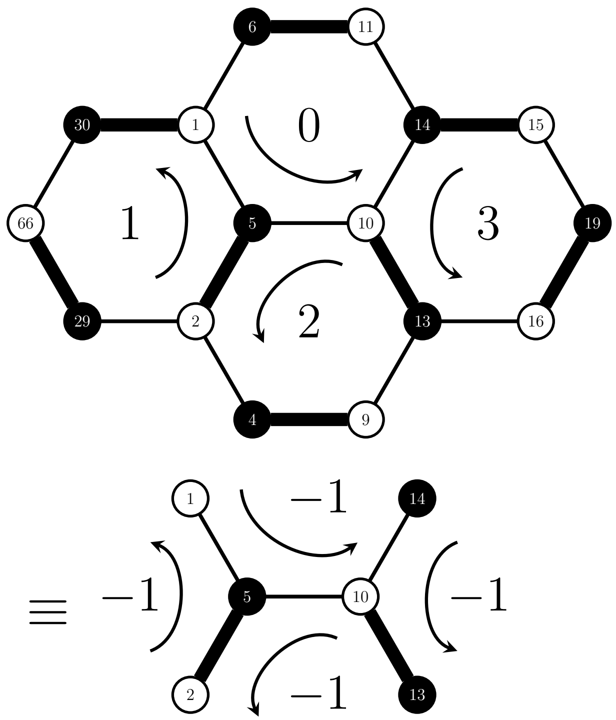
Reduce X to $\ll$ by $n \times n \longrightarrow \exp(\alpha n^2)$. Then starting at root, arbitrarily orient every k not crossing T^*. Deleting k^* from leaves, make $\varepsilon^{\mathscr{K}}_{\mathscr{F}}$, $\forall\, \mathscr{F}$.

Now,

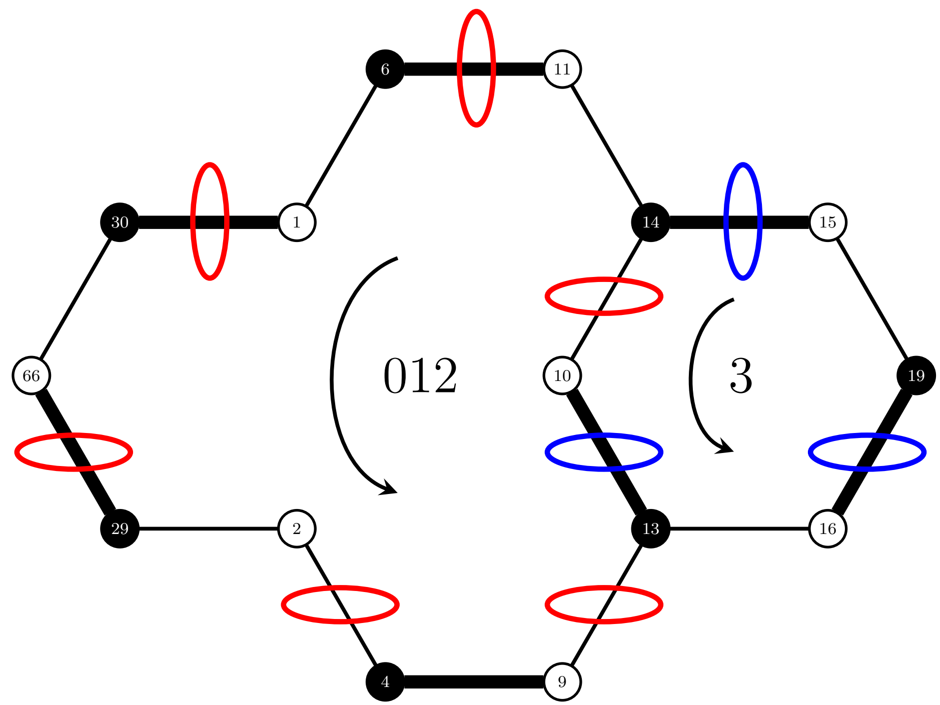
$$\prod_{\mathscr{F}\in\mathscr{F}_X}(-1)^{\rho^-_{\mathscr{F}}} = (-1)^{\left(\sum_{\mathscr{F}\in\mathscr{F}_X}\rho^-_{\mathscr{F}}\right)} = (-1)^{|\mathscr{E}_X|} \implies |\mathscr{V}_X| = \text{even}$$

by Euler-Poincaré $\sum_{\#}(-1)^{(\#\ \mathrm{mod}\ 2)}\,|\#\text{-cell}|$, $|\mathscr{F}_X| = |\mathscr{E}_X| - |\mathscr{V}_X| + 2g - 2$. □

Remark 2.1. Deleted-vertex "hole"-changes $\mathscr{K}$ to non-$\mathscr{K}$:

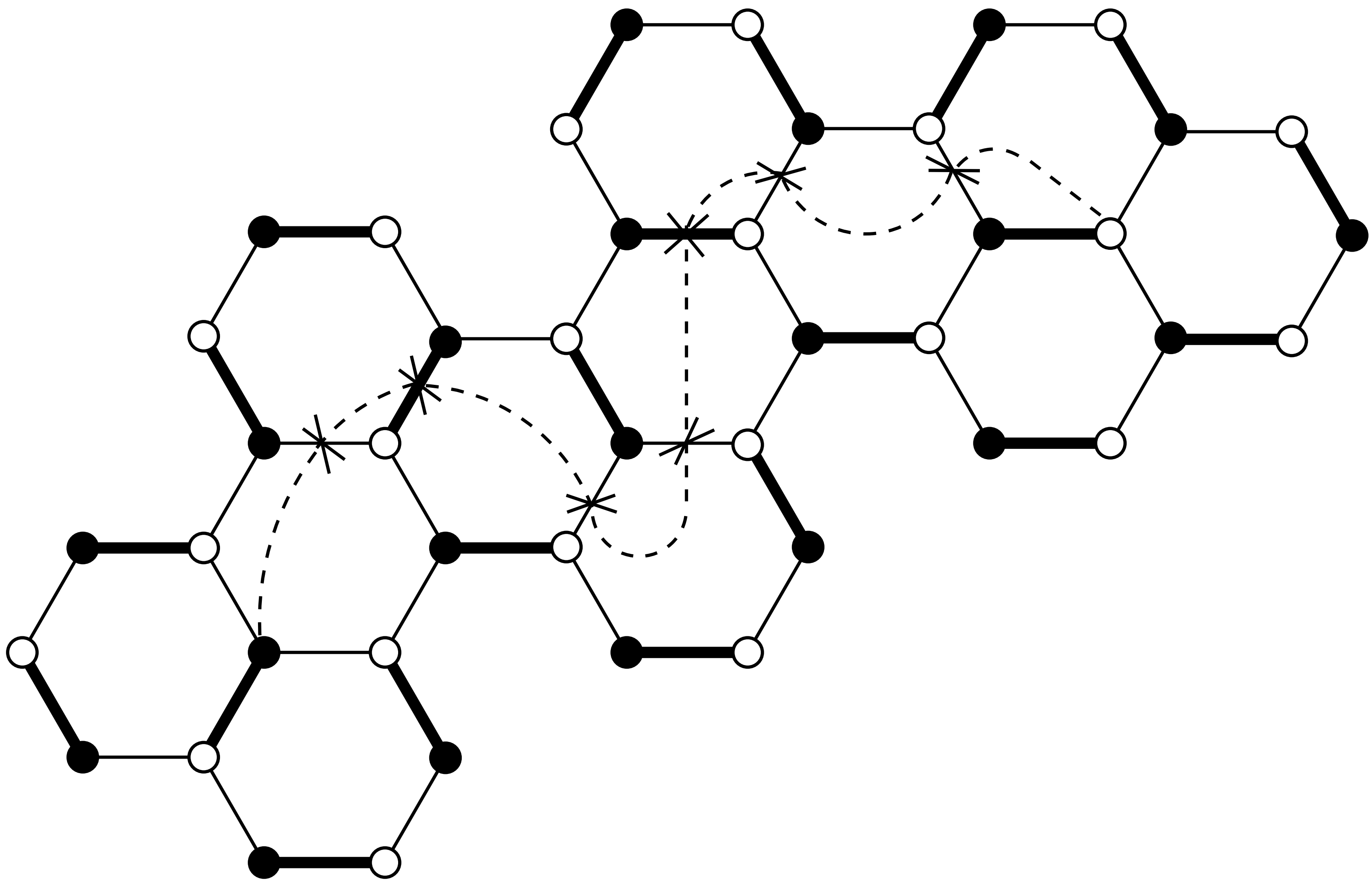


$h_0,\ h_1,\ h_2,\ h_3 = \boldsymbol{\mathscr{K}}$



$h_{012} =$ **non-$\mathscr{K}$**, $h_3 = \boldsymbol{\mathscr{K}}$.

Remark 2.2. To convert the non-$\mathscr{K}$ back to $\mathscr{K}$:



$$h_0 = h_1 = \cdots = h_{11} = -1.$$

Theorem 2.4. *Let* $X^{\mathscr{K}} \subset \overline{\mathcal{M}}_g \,|\, g=0$ *be a multiedge embedding, then*

$$\left|\mathsf{Pf}(X^{\mathscr{K}})\right| = \mathcal{Z} \stackrel{def}{=\!=} \sum_{D} \prod_{1=\mathbb{1}_{k|D}} \omega_k$$

where

$$\mathbf{Quot}(\mathbb{K}[D]) \ni \mathsf{Pf}(X^{\mathscr{K}}) = \frac{1}{(\frac{n}{2})!\; 2^{\frac{n}{2}}} \sum_{\sigma \in \mathcal{S}_n} sgn(\sigma)\, X^{\mathscr{K}}_{\sigma_1\sigma_2} \cdots X^{\mathscr{K}}_{\sigma_{n-1}\sigma_n}$$

$$= \frac{1}{a\Gamma(\frac{a}{b})\, b^{(\frac{a}{b}-1)}} \sum_{\sigma \subseteq \overline{[\sigma]} \in \{\overline{[\sigma]}\}} sgn(\sigma)\, X^{\mathscr{K}}_{\sigma_1\sigma_2} \cdots X^{\mathscr{K}}_{\sigma_{n-1}\sigma_n}$$

$$a = |Aut(\mathscr{D})|, \quad b = \Gamma\left(\frac{n}{2}+1\right)\Gamma^{n/2}(3)$$

$$sgn(\sigma) = (-1)^{t(\sigma)}, \quad t(\sigma) := (1\cdots n) \longleftarrow \sigma, \quad \frac{n}{2} \in \mathbb{Z}.$$

Proof. Following $\det X^{\mathcal{K}} = \det(-(X^{\mathcal{K}})^T) = (-1)^n \det X^{\mathcal{K}} = \mathsf{Pf}^2(X^{\mathcal{K}}) > 0$, i.e. nontrivial square rational function of positive semi-definite $(X^{\mathcal{K}}_{\xi\eta} \in \mathbb{R}^{n\times n})$ iff $\frac{n}{2} \in \mathbb{Z}$; in (Leibniz method) skew-symmetry monomials and partitions:

$$\sum_{\substack{\sigma = \widetilde{\sigma} \\ \{[\sigma]\} \ni \{\widetilde{\sigma}\} \\ \cong \\ \mathsf{Aut}(\mathcal{D})/(\mathcal{S}_{n/2} \times \mathcal{S}_2^{n/2})}} \Big(\prod_{\xi} X^{\mathcal{K}}_{\sigma_{2\xi-1}\,\sigma_{2\xi}} \Big)^2 \; + \; 2 \sum_{\substack{\{\sigma = \widetilde{\sigma}, \pi = \widetilde{\pi} \neq \widetilde{\sigma}\} \\ \widetilde{\sigma}, \widetilde{\pi} \neq \widetilde{\sigma} \in \{\widetilde{\sigma}\} \in \{[\sigma]\} \\ \cong \\ \mathsf{Aut}(\mathcal{D})/(\mathcal{S}_{n/2} \times \mathcal{S}_2^{n/2})}} (-1)^{t(\sigma) + t(\pi)} \prod_{\xi} X^{\mathcal{K}}_{\sigma_{2\xi-1}\,\sigma_{2\xi}} X^{\mathcal{K}}_{\pi_{2\xi-1}\,\pi_{2\xi}}$$

$$= \sum_{\sigma \in \mathcal{S}_n} (-1)^{t(\sigma)} \prod_{\xi} X^{\mathcal{K}}_{\xi\,\sigma_\xi} \; = \; \Big(\sum_{\sigma = \widetilde{\sigma}} \mathsf{sgn}(\sigma) \prod_{\xi} X^{\mathcal{K}}_{\sigma_{2\xi-1}\sigma_{2\xi}} \Big)^2 = \; \mathsf{Pf}^2(X^{\mathcal{K}})$$

where, for $\widetilde{\sigma}' = \sigma$: $(\sigma_{2\xi-1} > \sigma_{2\xi};\ \sigma_{2\xi} < \sigma_{2\xi+2})$,

$$t(\widetilde{\sigma}) \equiv t(\widetilde{\sigma}') \text{ for } \frac{n}{2} \in 2\mathbb{Z}, \quad t(\widetilde{\sigma}) \not\equiv t(\widetilde{\sigma}') \text{ for } \frac{n}{2} \in 2\mathbb{Z}+1$$

and,

$$\mathbb{1}_{2\mathbb{Z}\,|\,t(\widetilde{\sigma})} \; = \; 1 + \left\lfloor \frac{(n-1)!!}{2} \right\rfloor, \quad \mathbb{1}_{2\mathbb{Z}+1\,|\,t(\widetilde{\sigma})} \; = \; \left\lfloor \frac{(n-1)!!}{2} \right\rfloor.$$

That is, by

$$\mathsf{Pf}(X^{\mathscr{K}}) \;=\; \sum_{\sigma\,=\,\widetilde{\sigma}} \mathsf{sgn}(\sigma) \sum_{1\,=\,\mathbb{1}_{D|\sigma}} \;\prod_{1\,=\,\mathbb{1}_{k|D}} \varepsilon^{\mathscr{K}}_{k_{\sigma_{2\xi-1}\sigma_{2\xi}}} \omega_k \;=\; \pm\sqrt{\det X^{\mathscr{K}}}$$

and, by $\varepsilon^{\mathscr{K}}_D$ invariant of $\mathsf{Aut}(\mathscr{D})$, then

$$= \frac{1}{a\Gamma(\frac{a}{b})\, b^{(\frac{a}{b}-1)}} \sum_{\substack{\overline{[\sigma]}\,\in\,\{\overline{[\sigma]}\} \\ \cong \\ (\mathsf{Aut}(\mathscr{D})/(\mathcal{S}_{n/2}\times\mathcal{S}_2^{n/2})\times\cdots\times 1)\,\times\,(\mathcal{S}_{n/2}\times\mathcal{S}_2^{n/2})^{(\mathsf{Aut}(\mathscr{D})/(\mathcal{S}_{n/2}\times\mathcal{S}_2^{n/2}))\,\cong\,[\sigma]\,\cong\,\{\widetilde{\sigma}\}\,\cong\,\overline{[\sigma]}}}} \;\sum_{D\colon\,\sigma\,\subseteq\,\overline{[\sigma]}} \varepsilon^{\mathscr{K}}_D \prod_{1\,=\,\mathbb{1}_{k|D}} \omega_k$$

for all and any trivial $\mathcal{S}_n \backslash \mathsf{Aut}(\mathscr{D})$; thus, indeed,

$$= \frac{1}{(\frac{n}{2})!\; 2^{\frac{n}{2}}} \sum_{D\colon\,\sigma\,\in\,\mathsf{Aut}(\mathscr{D})} \varepsilon^{\mathscr{K}}_D \prod_{1\,=\,\mathbb{1}_{k|D}} \omega_k \;=\; \varepsilon^{\mathscr{K}}_D \sum_{D} \prod_{1\,=\,\mathbb{1}_{k|D}} \omega_k \;=\; \pm\,\mathcal{Z}$$

where $\varepsilon^{\mathscr{K}}_D = \pm 1$ only by orientation, independent of σ. Hence the claim. $\square$

Corollary 2.2. *For* $\det X^{\mathscr{K}} \mid X = K_n$, *the numbers* $\widetilde{\gamma}_n,\ \gamma_n^-,\ \gamma_n^+,\ \beta_n^+$ *of skew-annihilated-*, *unannihilated zero-*, *prior-to-skew-annihilation nonzero-*, *and unannihilated nonzero-monomials*, *respectively*:

$$\widetilde{\gamma}_n = \gamma_n^+ - \beta_n^+ = n! - \gamma_n^- - \beta_n^+;\ \ \beta_n^+ = \left(\frac{n!}{2^{\frac{n}{2}}\,\Gamma\left(\frac{n}{2}+1\right)}\right)^2 \ \left|\ \begin{array}{l}\beta_n^+, \gamma_n^-, \gamma_n^+ := 0 \\ \textit{if } \frac{n}{2} \notin \mathbb{Z}_{\geqslant 1}\end{array}\right.$$

$$\gamma_n^- = 1 + n! \sum_{s=2}^{n-1} (-1)^s \sum_{(\sigma_1,\ldots,\sigma_s)} \frac{\sigma_s! - 1}{\sigma_1! \cdots \sigma_s!} \ \left|\ \begin{array}{l} 1 \leqslant \sigma_1 \leqslant \cdots \leqslant \sigma_s \leqslant n-1 \\ 2 \leqslant s \leqslant n-1 \\ \sigma_1 + \cdots + \sigma_s = n \in \mathbb{Z}_{\geqslant 2}\end{array}\right.$$

$$= 1 + \Gamma(n+1) \sum_{s=2}^{n-1} (-1)^s \left(\frac{1}{\Gamma(s)} - \frac{1}{\Gamma(s)\ \Gamma(n+2-s)}\right).$$

Proof. Follows by bijection with the integer partition table of n. $\heartsuit$.

n	trivial	nontrivial partition $(\sigma_1, \ldots, \sigma_s)$ for γ_5^-						
5	$(5,0)$	$(1,4)$	$(2,3)$	$(3,2)$	$(1,1,3)$	$(1,2,2)$	$(2,1,2)$	$(1,1,1,2)$

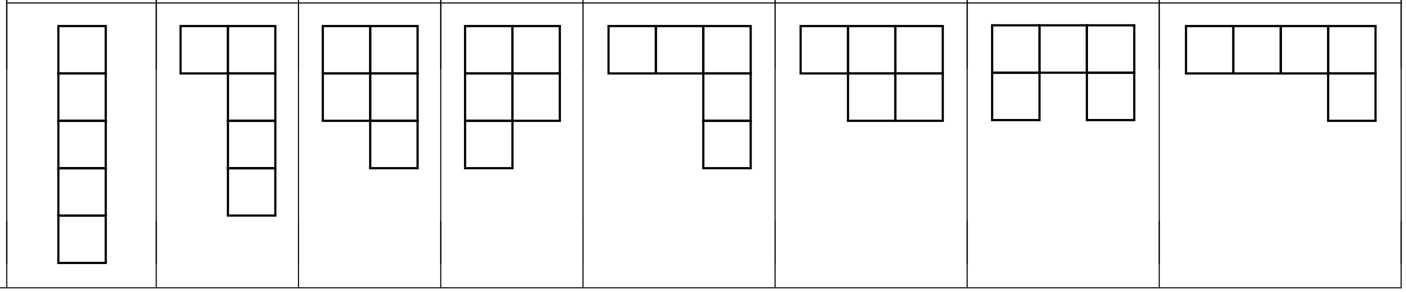


Table 1: Bijection integer partition and diagram for γ_5^-

$\widetilde{\gamma}_n = \gamma_n^+ - \beta_n^+$	γ_n^-	$\gamma_n^+ = n! - \gamma_n^-$	$\beta_n^+ \cong \{\widetilde{\sigma}\}\big\|_n$
0	1	1	1
0	15	9	9
40	455	265	225
3808	25487	14833	11025
441936	2293839	1334961	893025
68158816	302786759	176214841	108056025
13809632824	55107190151	32071101049	18261468225
3588233901120	13225725636255	7697064251745	4108830350625
1167849689703328	4047072044694047	2355301661033953	1187451971330625
466344469542546496	1537887376983737879	895014631192902121	428670161650355625
224453218323313949256	710503968166486900119	413496759611120779881	189043541287806830625
128246177964088857093664	392198190427900768865711	228250211305338670494289	100004033341249813400625
85860116510189002445897200	254928823778135499762712175	148362637348470135821287825	62502520838281133375390625
66597816144336476450233830048	192726190776270437820610404327	112162153835443422680893595673	45564337691106946230659765625
59261465838614835952392565344856	167671785975355280903931051764519	97581073836835777732377428235481	38319607998220941779984862890625
59975281959850766459952955571706496	166330411687552438656699603350402879	96800425246141091510518408809597121	36825143286290325050565453237890625

Table 2: Enumeration for $\frac{n}{2} = 1, \ldots, 16$

Theorem 2.5. *Observable is absolutely continuous iff $X^{\mathscr{K}}$ is non-singular.*

Proof. WLOG, by $D\cap(\sigma_1,\tau_1),\ldots,D\cap(\sigma_m,\tau_m)$, for $\mathcal{Z}=|\mathsf{Pf}(X^{\mathscr{K}})|$,

$$\left\langle \mathbb{1}_{\sigma_1\tau_1|D}\cdots\mathbb{1}_{\sigma_m\tau_m|D}\right\rangle = |\mathsf{Pf}\big((X^{\mathscr{K}})^{-1}\big)|\cdot|\mathsf{Pf}\big((X^{\mathscr{K}})_{ab}\big)|\Big|_{a,b\,\in\,\{\sigma_1,\tau_1,\ldots,\sigma_m,\tau_m\}}. \quad \square$$

Theorem 2.6. *Combinatorial exponential reduces to cubic complexity.*

Proof. $\mathsf{Pf}(\mathcal{A}\,X^{\mathscr{K}}\mathcal{A}^T)=\det(\mathcal{A})\mathsf{Pf}(X^{\mathscr{K}})\longrightarrow\mathcal{O}(n^3)$ in diagonalization by skew symmetric Gaussian elimination, for spectral analyses.

Remark. Recall critical point universality in mini-max contour deformation.

2.1 Graded (Grassmann) integral

Definition 2.2. $X^{\mathscr{K}}$ *basis* $(x_1,\dots,x_n)$ *graded (Grassmann) algebra* $\bigwedge^{\star} X^{\mathscr{K}}$:

$$\left\{\begin{matrix} x_0=1;\ x_{\sigma_k<}=x_{\sigma_1}\otimes\cdots\otimes x_{\sigma_k} \\ \forall\,\sigma_k\in\{1,\dots,n\};\ \ k=1,\dots,n \end{matrix}\ \middle|\ \begin{matrix} x_{\sigma_\xi}\otimes x_{\sigma_\eta}+x_{\sigma_\eta}\otimes x_{\sigma_\xi}=0 \\ \sigma_k< \Longrightarrow(\sigma_1,\dots,\sigma_k)\mid \sigma_1<\cdots<\sigma_k \end{matrix}\right\}.$$

Thus, element is graded:

$$\bigwedge^{\star} X^{\mathscr{K}} \ni y(x) \;=\; y^{(0)} \;\oplus\; \sum_{a=1}^{n} y^{(a)} x_a \;\oplus\; \bigoplus_{k=2}^{n} \sum_{\tau\in\mathcal{S}_k} (-1)^{t(\tau)}\, y^{(\tau_1,\dots,\tau_k)}\, x_{\tau_k<}$$

$$= \bigoplus_{k=0}^{n} \sum_{\tau_k\in\mathcal{S}_k} y^{(\tau_k)} \bigotimes_{a=1}^{k} x_{\tau_a} \ \left|\ \begin{matrix} x_{\tau_0}=x_0=1,\ \ y^{(\tau_0)}=y^{(0)} \\ \mathcal{S}_o=1,\ \ \tau_{k\geqslant 1}=(\tau_1,\dots,\tau_k). \end{matrix}\right.$$

For multiplication: $(\sigma_1\wedge\cdots\wedge\sigma_n)\wedge(\tau_1\wedge\cdots\wedge\tau_k) = \sigma_1\wedge\cdots\wedge\sigma_n\wedge\tau_1\wedge\cdots\wedge\tau_k$;

$$y_1(x)\,y_2(x) = y_1^{(0)}y_2^{(0)} \oplus \sum_{a=1}^{n}\big(y_1^{(0)}y_2^{(a)} + y_1^{(a)}y_2^{(0)}\big)x_a \oplus \frac{1}{2}\sum_{\sigma\in\mathcal{S}_2}\big(y_1^{(0)}y_2^{(\sigma_1\sigma_2)} +$$

$$+\, y_1^{(\sigma_1)}y_2^{(\sigma_2)} - y_1^{(\sigma_2)}y_2^{(\sigma_1)} + y_1^{(\sigma_1\sigma_2)}y_2^{(0)}\big)x_{\sigma_1}\otimes x_{\sigma_2} \oplus \cdots$$

Remark 2.3. That is, $\dim \bigwedge^{\star} X^{\mathscr{K}} = 2^n = \sum_{k=0}^{n} \dim \bigwedge^{k} X^{\mathscr{K}} = \sum_{k=0}^{n} \binom{n}{k}$.

Derivation 2.1. $\bigwedge^n X^{\mathscr{K}} \ni w^{\frac{n}{2}} = \mathsf{Pf}(X^{\mathscr{K}}) x_{\sigma_n <}$; $\bigwedge^2 X^{\mathscr{K}} \ni w = \sum_{ab} X^{\mathscr{K}}_{ab} x_a \otimes x_b$;

$$\bigotimes^k X^{\mathscr{K}} \longrightarrow \bigotimes^k X^{\mathscr{K}}: \ (-1)^{t(\sigma)} w_{\sigma_1} \wedge \cdots \wedge w_{\sigma_k} = \frac{1}{k!} \sum_{\tau \in \mathcal{S}_{\sigma_k <}} (-1)^{t(\tau)} \bigotimes_{a=1}^{k} w_{\tau_a}.$$

Definition 2.3. *With respect to orientation* $x \in \bigwedge^n X^{\mathscr{K}} \cong \mathbb{R}$,

$$\int_{\bigwedge^\star X^{\mathscr{K}}} f = \int_{\bigwedge^n X^{\mathscr{K}}} f = f_x \ \Big| \ f = f_x\, x + \underbrace{\cdots}_{\text{lower order terms}}$$

by formal rule

$$\int \bigotimes_{a=1}^{n} x_a \otimes \bigotimes_{a=1}^{n} dx_a = (-1)^{\left(\sum_{a=1}^{n-1} a\right)} \int \bigotimes_{a=1}^{n} (x_a \otimes dx_a) = (-1)^{\frac{n}{2}(n-1)}.$$

Derivation 2.2. For degenerate integral if $\deg(x) < \deg(dx)$,

$$\int \bigotimes_{a=1}^{k} x_{\sigma_a} \otimes dx = \begin{cases} (-1)^{t(\sigma)} & \text{if } k = n \\ 0 & \text{if } k < n \end{cases} \ \Bigg| \ \begin{aligned} & dx = (-1)^{\frac{n}{2}(n-1)} \textstyle\bigotimes_{a=1}^{n} dx_a \\ & t(\sigma) := \sigma = (\sigma_1 \cdots \sigma_n) \longrightarrow (1 \cdots n). \end{aligned}$$

Derivation 2.3. $\theta = x_1 \otimes \cdots \otimes x_n$ is well-defined if (x_a) is basis of $X^{\mathscr{K}}$.

Theorem 2.7. *Let* $f(x) = \int_{\bigwedge^\star X^{\mathscr{K}}} \exp\Big(\lambda_0 + \frac{1}{2}\left\langle x, X^{\mathscr{K}} x\right\rangle\Big)\,dx$ *satisfy* $\mathsf{Pf}(X^{\mathscr{K}})$ *constraints; then* f *uniquely maximizes* $-\int_{\bigwedge^\star X^{\mathscr{K}}} (1/|f|)\log(1/|f|)\,dx$; *and,*

(i) $$\mathsf{Pf}(X^{\mathscr{K}}) = \int_{\bigwedge^\star X^{\mathscr{K}}} \exp\Big(\frac{1}{2}\sum_{ab} x_a X^{\mathscr{K}}_{ab} x_b\Big)\,dx$$

(ii) $$\mathsf{Pf}\begin{pmatrix} 0 & X^{\mathscr{K}} \\ -(X^{\mathscr{K}})^T & 0 \end{pmatrix} = \det(X^{\mathscr{K}})$$

(iii) $$(\mathsf{Pf}(X^{\mathscr{K}}))^2 = \det(X^{\mathscr{K}})$$

(iv) $$\frac{\partial}{\partial X^{\mathscr{K}}_{a_1 b_1}} \cdots \frac{\partial}{\partial X^{\mathscr{K}}_{a_k b_k}} \mathsf{Pf}(X^{\mathscr{K}}) = \mathsf{Pf}(X^{\mathscr{K}}) \cdot \mathsf{Pf}((X^{\mathscr{K}^{-1}})_{xy})\Big|_{\substack{x = a_1, \ldots, a_k \\ y = b_1, \ldots, b_k}}.$$

Proof.

(i). Since all exponent, except $(\frac{n}{2})$th, vanishes, then

$$\int_{\bigwedge^{\star} X^{\mathscr{K}}} \exp\Big(\frac{1}{2}\left\langle x, X^{\mathscr{K}} x\right\rangle\Big) dx \;=\; \frac{1}{(\frac{n}{2})!}\,\frac{1}{2^{\frac{n}{2}}} \int_{\bigwedge^{\star} X^{\mathscr{K}}} \left\langle x, X^{\mathscr{K}} x\right\rangle^{\frac{n}{2}} dx$$

where

$$\int \left\langle x, X^{\mathscr{K}} x\right\rangle^{\frac{n}{2}} dx \;= \int \sum_{\sigma \in \mathcal{S}_{n/2} \times \mathcal{S}_2^{n/2}} X^{\mathscr{K}}_{a_1 b_1} \cdots X^{\mathscr{K}}_{a_n b_n} (x_{a_1} \otimes x_{b_1}) \otimes \cdots \otimes (x_{a_n} \otimes x_{b_n})\, dx$$

$$= \sum_{\sigma \in \mathcal{S}_{n/2} \times \mathcal{S}_2^{n/2}} (-1)^{t(\sigma)} X^{\mathscr{K}}_{a_1 b_1} \cdots X^{\mathscr{K}}_{a_n b_n} \;\Big|\; t(\sigma) := \sigma = (a_1 b_1 \cdots a_n b_n) \longrightarrow (1 \cdots n)$$

that is, by "equality" for permutations $\sigma \in \mathcal{S}_{n/2} \times \mathcal{S}_2^{n/2}$, then

$$\int_{\bigwedge^{\star} X^{\mathscr{K}}} \exp\Big(\frac{1}{2}\left\langle x, X^{\mathscr{K}} x\right\rangle\Big) dx \;= \mathsf{Pf}(X^{\mathscr{K}}).$$

Hence, we are done; moreover II, III and IV follow by the latter integral. $\square$

(ii). Choosing splitting $X^{\mathscr{K}} = W^{\mathscr{K}} \oplus W^{\mathscr{K}}$ for block structure, where $X^{\mathscr{K}}$ is isomorphic to algebra (tensor product) generated by $u_a, v_a \,|\, a=1,\ldots,\frac{n}{2}$ with relations $u_a u_b = -u_b u_a$, $u_a v_b = -v_b u_a$, and $v_a v_b = -v_b v_a$:

$$\begin{aligned}(x_1,\ldots,x_n) &= \\ &= \big(\underbrace{u_1,\ldots,u_{\frac{n}{2}}}_{\text{basis in } W^{\mathscr{K}}}, \underbrace{v_1,\ldots,v_{\frac{n}{2}}}_{\text{basis in } W^{\mathscr{K}}}\big).\end{aligned}$$

As a result,

$$\left\langle x\,,\ \begin{pmatrix} 0 & X^{\mathscr{K}} \\ -(X^{\mathscr{K}})^T & 0 \end{pmatrix} x \right\rangle \;=\; 2\,\langle u, X^{\mathscr{K}} v\rangle$$

i.e. need to prove

$$\int_{\bigwedge^n (W^{\mathscr{K}} \oplus W^{\mathscr{K}})} \exp\left(\langle u, X^{\mathscr{K}} v\rangle\right) du\, dv \;=\; \det(X^{\mathscr{K}}).$$

(iii). Similar.

(iv). $$\int \exp\Big(\frac{1}{2}\langle x, X^{\mathcal{K}}x\rangle + \langle x, \eta\rangle\Big)dx \quad =$$

$$= \int \exp\Big(\frac{1}{2}\Big\langle x + X^{\mathcal{K}\,-1}\eta\,,\; X^{\mathcal{K}}(x + X^{\mathcal{K}\,-1}\eta)\Big\rangle \;-\; \frac{1}{2}\Big\langle \eta, X^{\mathcal{K}\,-1}\eta\Big\rangle\Big)dx$$

$$= \exp\Big(-\frac{1}{2}\Big\langle \eta, X^{\mathcal{K}\,-1}\eta\Big\rangle\Big)\,\mathsf{Pf}(X^{\mathcal{K}}).$$

$$\frac{\partial}{\partial X^{\mathcal{K}}{}_{a_1 b_1}} \cdots \frac{\partial}{\partial X^{\mathcal{K}}_{a_k b_k}}\; \mathsf{Pf}(X^{\mathcal{K}}) \;=$$

$$= \int \exp\Big(\frac{1}{2}\langle x, X^{\mathcal{K}}x\rangle\Big)\, x_{a_1} x_{b_1} \cdots x_{a_k} x_{b_k}\, dx$$

$$= \Big(\frac{\partial}{\partial \eta}\Big)^{2k} \int \exp\Big(\frac{1}{2}\langle x, X^{\mathcal{K}}x\rangle + \langle \eta, x\rangle\Big)dx.$$

Theorem 2.8. *Let* $X^{\mathscr{K}} \subset \overline{\mathcal{M}}_g \mid g=0$ *be bipartite multiedge embedding, then*

(i) $\mathcal{Z} = |\det(C_{X^{\mathscr{K}}})| \;\; \Big| \, C_{X^{\mathscr{K}}} = \mathbb{R}^{\mathscr{V}^{\circ}_{X^{\mathscr{K}}}} \hookleftarrow; \;\; \mathbb{R}^{\mathscr{V}_{X^{\mathscr{K}}}} = \mathbb{R}^{\mathscr{V}^{\bullet}_{X^{\mathscr{K}}}} \oplus \mathbb{R}^{\mathscr{V}^{\circ}_{X^{\mathscr{K}}}} \hookleftarrow$

where $\hookleftarrow \implies$ *nested.*

(ii) $\left\langle \boldsymbol{\sigma}_{a_1 b_1} \cdots \boldsymbol{\sigma}_{a_k b_k} \right\rangle = \det\!\big((C_{X^{\mathscr{K}}})^{-1}\big) \det\!\big((C_{X^{\mathscr{K}}})_{\widetilde{a}\, b}\big) \;\; \left| \begin{array}{l} \widetilde{a} = (\widetilde{a}_1, \ldots, \widetilde{a}_k) \\ b = (b_1, \ldots, b_k) \end{array} \right.$

where $\widetilde{a} =$ *white-vertex identified with* a.

Proof.

(i). $X^{\mathscr{K}} \subset \overline{\mathcal{M}}_g \mid g=0$ implies

$$\mathcal{Z} \;=\; \boldsymbol{\varepsilon}_X^{\mathscr{K}} \int \exp\Big(\frac{1}{2}\sum_{ab} x_a\,(X^{\mathscr{K}}_{ab})\,x_b\Big)dx \qquad \left|\begin{array}{l} \boldsymbol{\varepsilon}_X^{\mathscr{K}} = (-1)^{\boldsymbol{\sigma}}\,\boldsymbol{\varepsilon}^{\mathscr{K}}_{\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2}\cdots\boldsymbol{\varepsilon}^{\mathscr{K}}_{\boldsymbol{\sigma}_{n-1}\boldsymbol{\sigma}_n} \\ n = |V(X^{\mathscr{K}})|. \end{array}\right.$$

$X^{\mathscr{K}} \subset \overline{\mathcal{M}}_g \mid g=0$ bipartite $\mathscr{V}_{X^{\mathscr{K}}} = \mathscr{V}^{\bullet}_{X^{\mathscr{K}}} \sqcup \mathscr{V}^{\circ}_{X^{\mathscr{K}}}$ implies

$$X^{\mathscr{K}} \;=\; \begin{pmatrix} 0 & B_{X^{\mathscr{K}}} \\ -(B_{X^{\mathscr{K}}})^T & 0 \end{pmatrix} \qquad \left|\begin{array}{l} B_{X^{\mathscr{K}}} :\; \mathbb{R}^{\mathscr{V}^{\circ}_{X^{\mathscr{K}}}} \longrightarrow \mathbb{R}^{\mathscr{V}^{\bullet}_{X^{\mathscr{K}}}} \\ \mathbb{R}^{V(X^{\mathscr{K}})} = \; \mathbb{R}^{\mathscr{V}^{\bullet}_{X^{\mathscr{K}}}} \oplus \mathbb{R}^{\mathscr{V}^{\circ}_{X^{\mathscr{K}}}} \\ \dim(\mathbb{R}^{\mathscr{V}^{\bullet}_{X^{\mathscr{K}}}}) = \dim(\mathbb{R}^{\mathscr{V}^{\circ}_{X^{\mathscr{K}}}}) = \frac{n}{2} \\ |V(X^{\mathscr{K}})| = n. \end{array}\right.$$

Identifying $V_{\bullet}(X^{\mathscr{K}})$, $V_{\circ}(X^{\mathscr{K}})$, via a diagram $\{b\} \sim \{w\}$ with “hole”

$$X^{\mathscr{K}} \;=\; \begin{pmatrix} 0 & C_{X^{\mathscr{K}}} \\ -(C_{X^{\mathscr{K}}})^T & 0 \end{pmatrix} \qquad \left|\begin{array}{l} \mathbb{R}^{V(X^{\mathscr{K}})} = \; \mathbb{R}^{\mathscr{V}^{\bullet}_{X^{\mathscr{K}}}} \oplus \mathbb{R}^{\mathscr{V}^{\circ}_{X^{\mathscr{K}}}} \hookleftarrow \\ C_{X^{\mathscr{K}}} \;=\; \mathbb{R}^{\mathscr{V}^{\circ}_{X^{\mathscr{K}}}} \hookleftarrow \\ \text{where } \hookleftarrow \;\Longrightarrow\; \text{nested} \end{array}\right.$$

i.e. $\mathcal{Z} \;=\; |\det(C_{X^{\mathscr{K}}})|$. □

(ii). Write

$$\left\langle \boldsymbol{\sigma}_{a_1 b_1} \cdots \boldsymbol{\sigma}_{a_k b_k} \right\rangle = \frac{\partial}{\partial\, w(a_1 b_1)} \cdots \frac{\partial}{\partial\, w(a_k b_k)} \ \ln \mathcal{Z}$$

$$= \det\!\big((C_{X\mathfrak{K}})^{-1}\big) \det\!\big((C_{X\mathfrak{K}})_{\widetilde{a}\, b}\big) \ \ \Big|\begin{array}{l} \widetilde{a} = \widetilde{a}_1, \ldots, \widetilde{a}_k \\ b = b_1, \ldots, b_k \end{array}$$

where $\widetilde{a} =$ white-vertex identified with a. $\square$

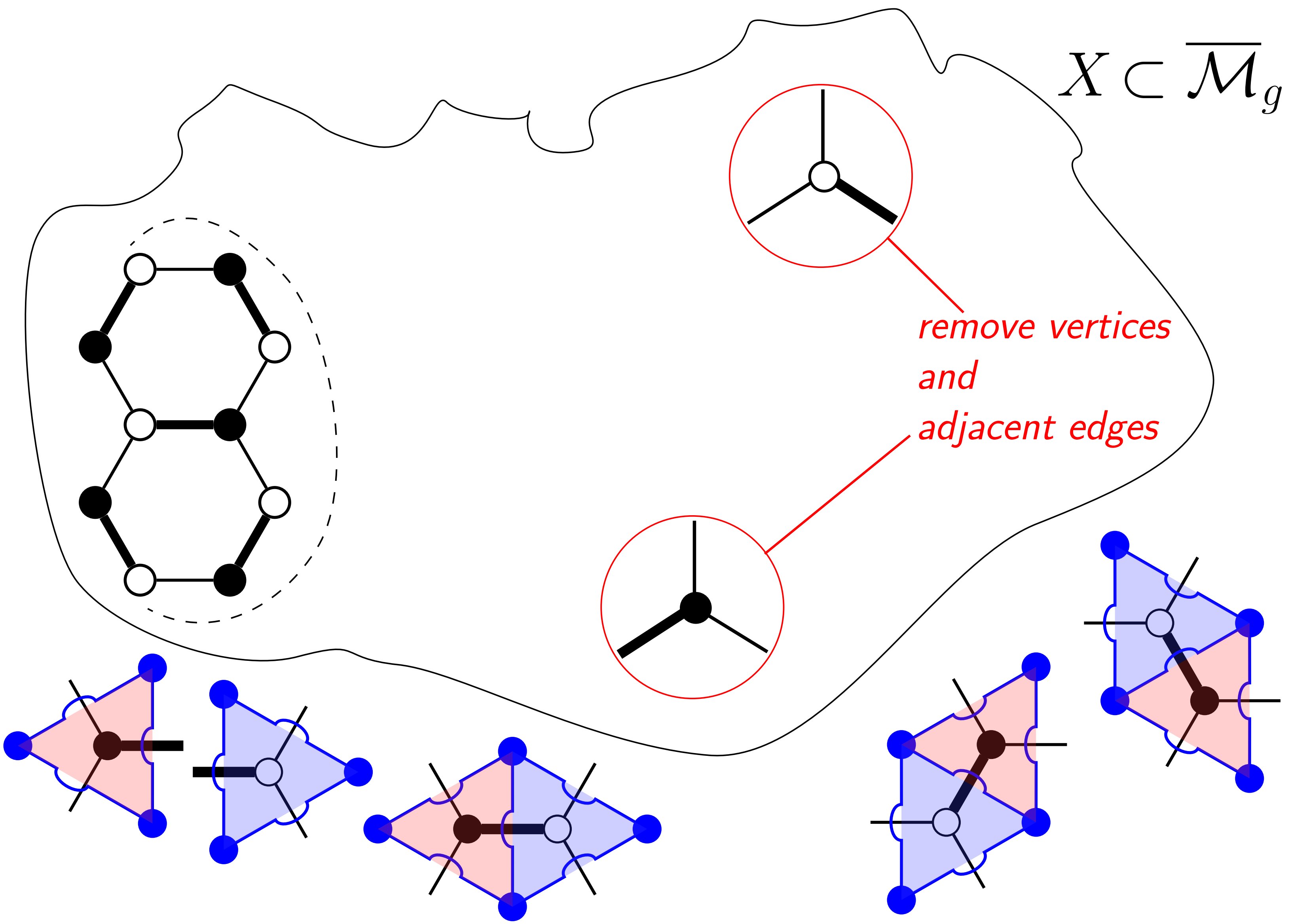
Remark. The "physical" meaning:

$$\left\langle \boldsymbol{\sigma}_{a_1 b_1} \cdots \boldsymbol{\sigma}_{a_k b_k} \right\rangle =$$

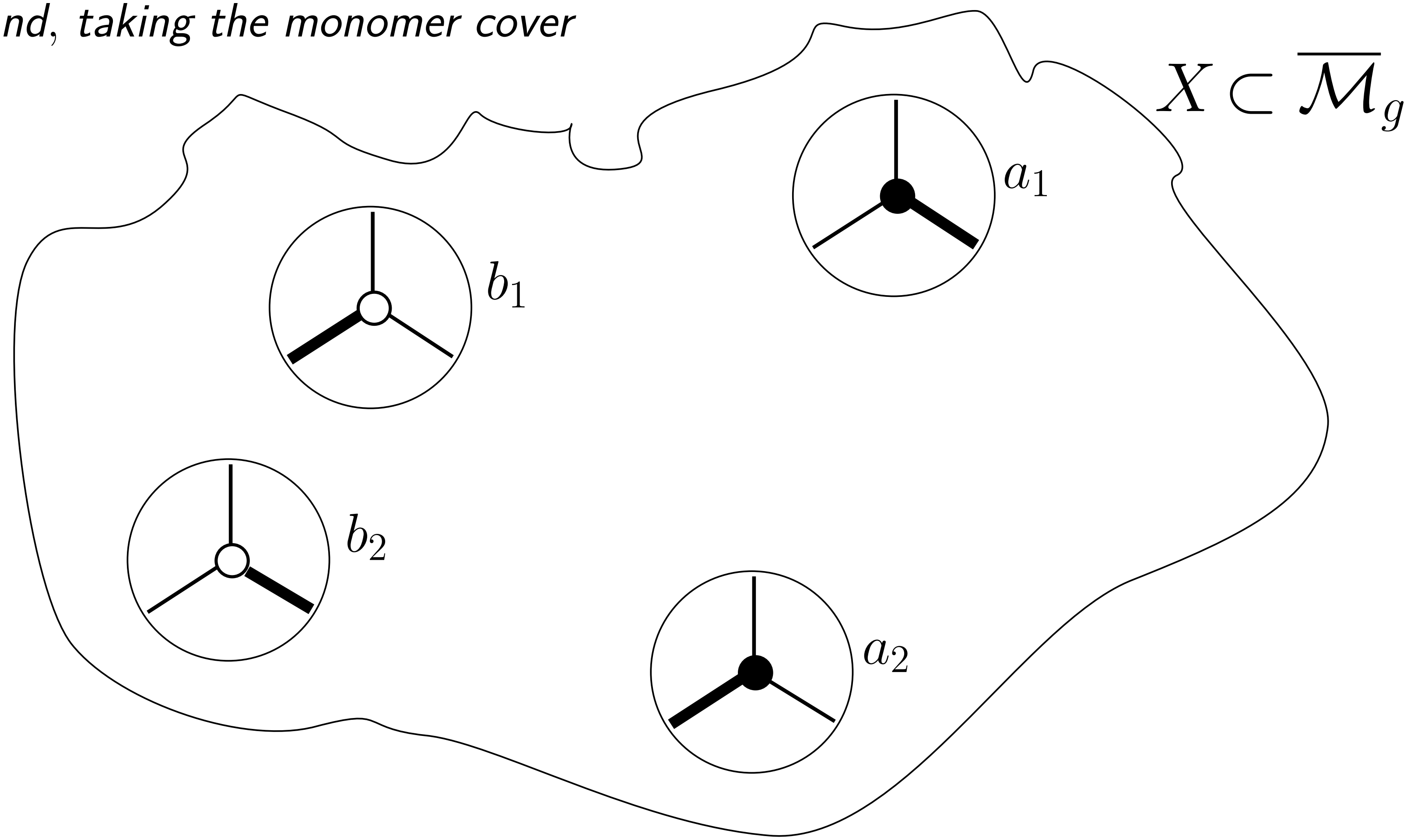
$$= \int \psi^*_{a_1} \psi_{b_1} \cdots \psi^*_{a_k} \psi_{b_k} \exp\!\big(\psi^* C_{X\mathfrak{K}}\, \psi\big) d\psi^* d\psi \ \cdot \int \exp\!\big(\psi^* C_{X\mathfrak{K}}\, \psi\big) d\psi^* d\psi$$

which corresponds to the free Fermionic observable.

Corollary 2.3 (monomer problem). *Given monomers $\longleftrightarrow$ dimers:*



and, taking the monomer cover

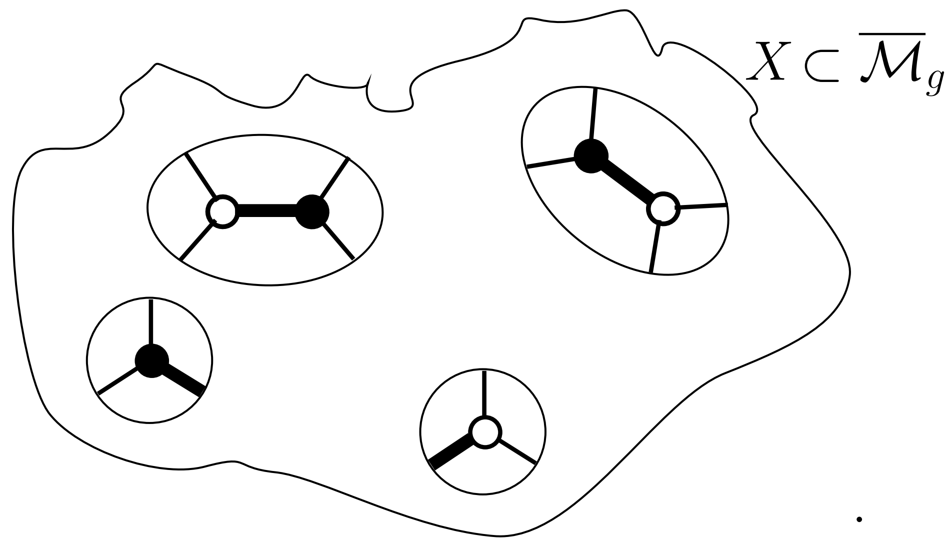


then, monomer-monomer observable

$$M_{a_1 \cdots a_n b_1 \cdots b_n} = \frac{\mathcal{Z}(X^{\mathscr{K}}_{a_1 \cdots a_n b_1 \cdots b_n})}{\mathcal{Z}(X^{\mathscr{K}})}.$$

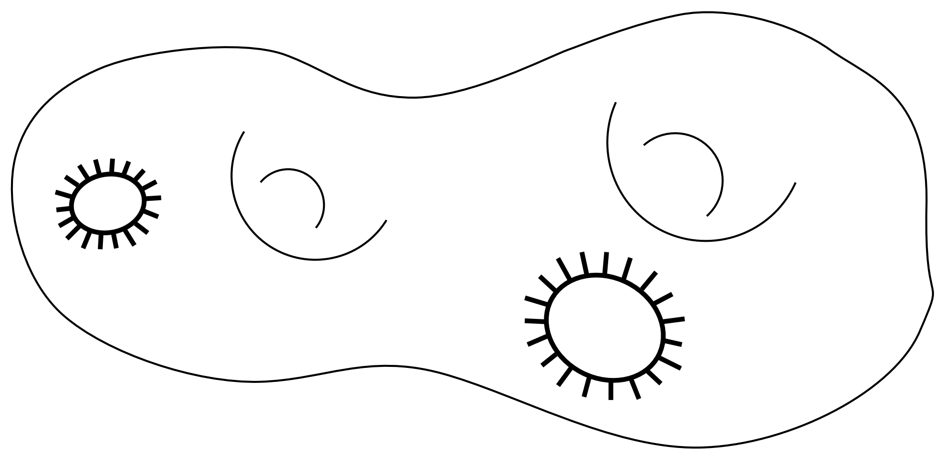
Proof. $\heartsuit$.

Remark 2.4. Exactly, adjacent monomers a_ξ and b_ξ give dimer $\mathbb{1}_{\xi|D} = 1$:



.

Moreover, $M_{a_1 \cdots a_n b_1 \cdots b_n}$ for all $|\{[\mathscr{K}]\}| = 2^{2g+n-1}$, $n = |\partial D|$, is special case for the nontrivial fundamental-group surfaces:



.

2.2 Partition as sum of Pfaffians

Lemma 2.1 (R. et al., 2005).

$$\mathcal{Z} = \frac{1}{2^g} \sum_{\{[\mathcal{K}]\}} \mathsf{Arf}(q_D^{\mathcal{K}}) \cdot \varepsilon^{\mathcal{K}}(D) \cdot \mathsf{Pf}(X^{\mathcal{K}}) \qquad \left| \begin{array}{l} \pm 1 = \mathsf{Arf}(q) = \dfrac{1}{2^g} \displaystyle\sum_{\alpha \in \mathcal{H}^1} (-1)^{q(\alpha)} \\ 2^g = |\mathcal{H}^1(X^{\mathcal{K}};\, \mathbb{Z}_2)| \end{array} \right.$$

where

$\{[\mathcal{K}]\} =$ *all equivalence classes of* $\mathcal{K}$, 2^{2g} *in total*

$q_D^{\mathcal{K}} =$ *quadratic form on* $\mathcal{H}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2)$,
corresponding to $\mathcal{K}$ *with respect to a reference* $D \subseteq \mathcal{D}$

$$\varepsilon^{\mathcal{K}}(D) = (-1)^{\sigma}\, \varepsilon^{\mathcal{K}}_{\sigma_1\sigma_2} \cdots \varepsilon^{\mathcal{K}}_{\sigma_{n-1}\sigma_n} \qquad \left| \begin{array}{l} \sigma \in \mathit{Aut}(D) \subseteq \mathit{Aut}(\mathcal{D}) \subseteq \mathcal{S}_n \\ \mathcal{D} \cong [\sigma] \cong \mathit{Aut}(\mathcal{D}) \big/ (\mathcal{S}_{n/2} \times \mathcal{S}_2^{n/2}). \end{array} \right.$$

Proof. $\heartsuit$.

Theorem 2.9 (R. et al., 2005).

$$\mathcal{Z} = \frac{1}{2^g} \sum_{\mathfrak{T} \in S(\overline{\mathcal{M}}_g)} \mathsf{Arf}(q^{\mathscr{K}}_{\mathfrak{T}}) \cdot \mathsf{Pf}(X^{\mathscr{K}}_{\mathfrak{T}}) \quad \left| \begin{array}{l} \pm 1 = \mathsf{Arf}(q) = \frac{1}{2^g} \sum_{\alpha \in \mathcal{H}^1} (-1)^{q(\alpha)} \\ 2^g = |\mathcal{H}^1(X^{\mathscr{K}};\, \mathbb{Z}_2)| \end{array} \right.$$

where

$\mathsf{Arf}(q^{\mathscr{K}}_{\mathfrak{T}}) :=$ *quadratic form* $q^{\mathscr{K}}_{\mathfrak{T}}$ *on* $\mathcal{H}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2)$ *for spin structure* $\mathfrak{T}$

$X^{\mathscr{K}}_{\mathfrak{T}} = \mathscr{K}$ *matrix corresponding to spin structure* $\mathfrak{T}$

$S(\overline{\mathcal{M}}_g) =$ *set of all spin structures on* $\overline{\mathcal{M}}_g$.

Proof. $\heartsuit$.

Theorem 2.10 (R. et al., 2005). *Let* $X \subset \overline{\mathcal{M}}_g$ *be bipartite, such that*

height function $=$

$=$ *section of the non-trivial* $\mathbb{Z}$*-bundle*

then

$$\mathcal{Z}\big(\mathscr{H}_{x_1}, \ldots, \mathscr{H}_{x_g}, \mathscr{H}_{y_1}, \ldots, \mathscr{H}_{y_g}\big) =$$

$$= \sum_D \prod_{\ell \in D} \boldsymbol{\omega}(\ell) \prod_{\xi=1}^{g} \exp\Big(\sum_{\xi} \mathscr{H}_{x_\xi} \Delta_{x_\xi} h \ + $$
$$+ \sum_{\xi} \mathscr{H}_{y_\xi} \Delta_{y_\xi} h \Big)$$

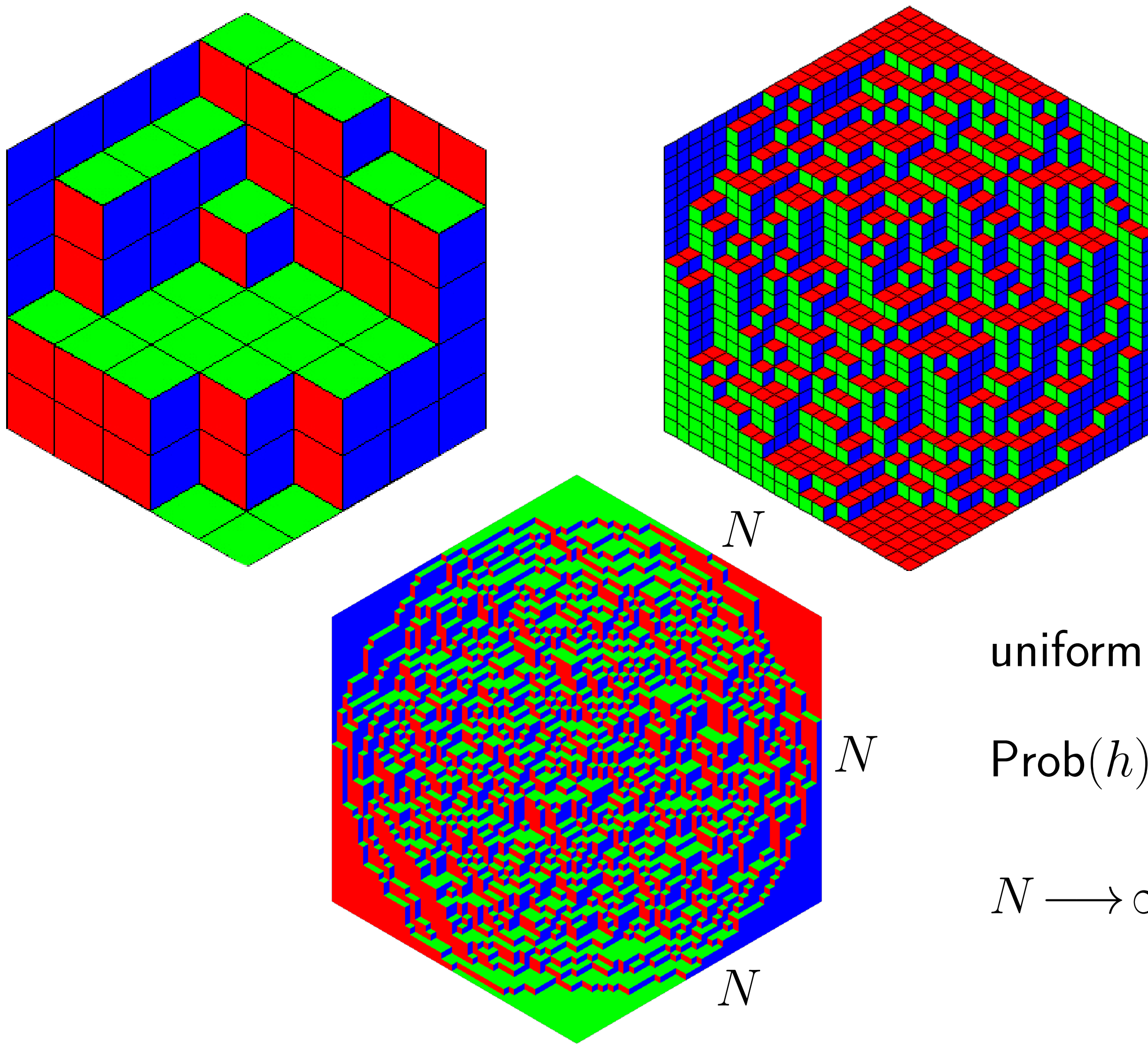
where

$(x_1, \ldots, x_g, y_1, \ldots, y_g) =$ *fundamental cycles*

$\Delta_C h =$ *change in height function along* $\overline{\mathcal{M}}_g$ *noncontractible cycle* C.

Proof. $\heartsuit$.

2.3 Limits

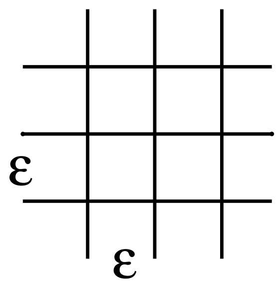


uniform measure

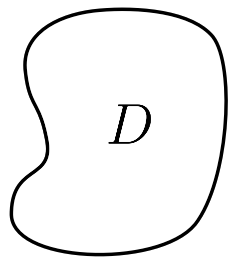
$$\text{Prob}(h) = \frac{1}{|\mathcal{H}_X|}$$

$$N \longrightarrow \infty\,.$$

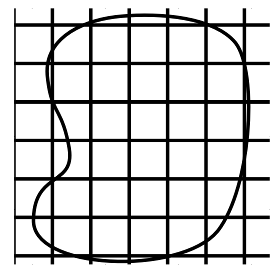
Theorem 2.11 (Schur; Okounkov & R). *Let* $\varphi_\varepsilon : \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2 \,|\, D \subset \mathbb{R}^2$;



;



;



;

such that
$\varepsilon \longrightarrow 0$, *as* $|D_\varepsilon| \longrightarrow \infty$
with $D_\varepsilon = D \cap \varphi_\varepsilon(\mathbb{Z}^2)$

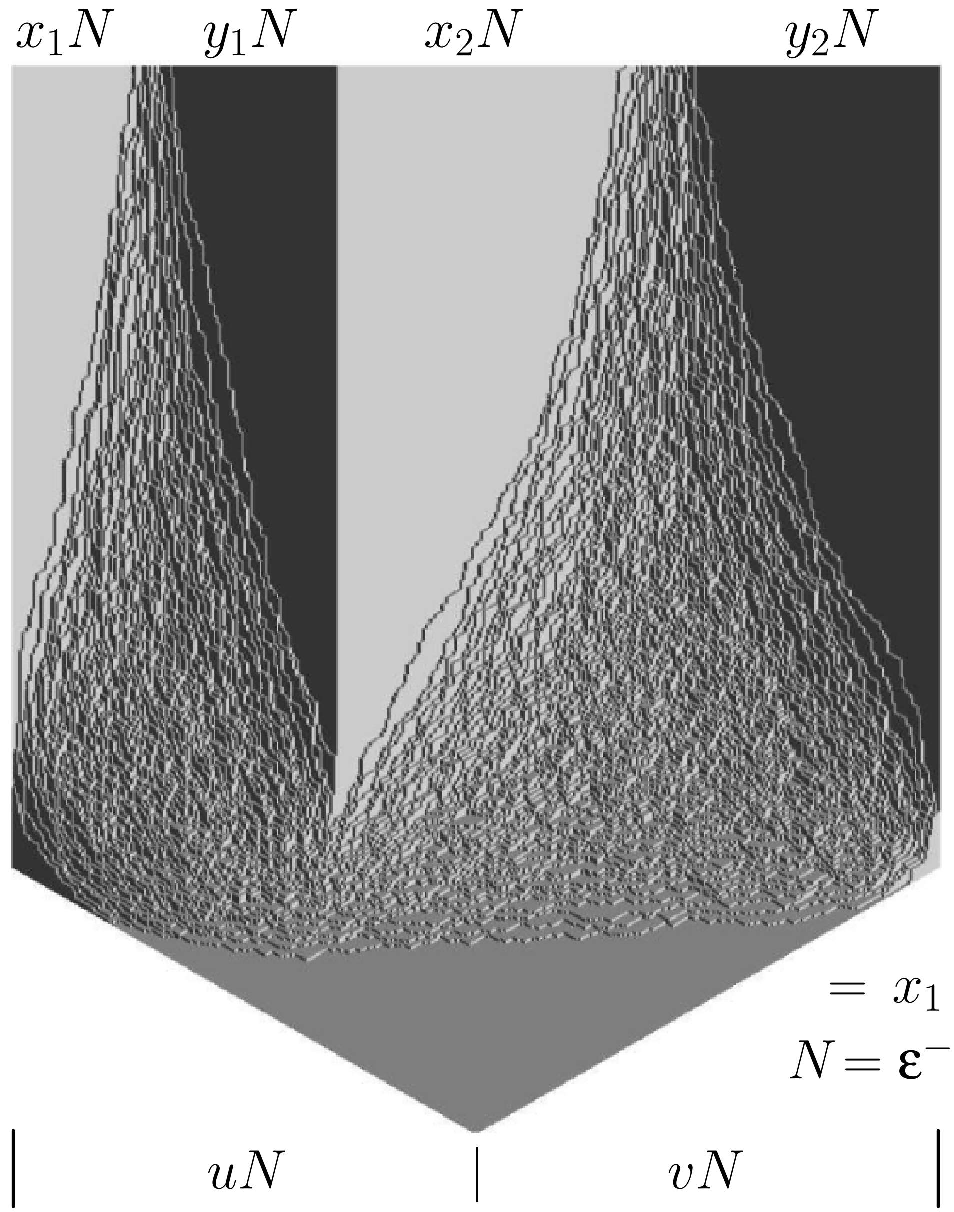
for cube-stack measure

$$\textit{Prob}(\pi) = \frac{\prod_t q_t^{\pi(t)}}{\sum_\pi \prod_t q_t^{\pi(t)}} \quad \left| \begin{array}{l} \pi \in \mathcal{H}_X \\ \pi \cong D \end{array} \right.$$

then there is existence of:

$$\textit{Thermodynamic limit } (|D_\varepsilon| \longrightarrow \infty) \; + \\ + \; \textit{Scaling limit } (q = e^{-\varepsilon}, \; \varepsilon \longrightarrow +0).$$

Proof. $\heartsuit$.



where $u + v =$
$= x_1 + x_2 + y_1 + y_2$;
$N = \varepsilon^{-1}$, $q = e^{-\varepsilon}$.

3 Vertex algebras

Points:

(i) Reformulate Kasteleyn Grassmann integral on transfer matrices Pf$(\cdot)$

(ii) Prove Grassmann kernel convergence for T^* special genus g domain

(iii) Obtain $\mathbb{R}$ log scaling asymptotics, limit shape variational principle

(iv) State the conjecture for Green's function $\langle \cdot \rangle$ in large deviation

3.1 Graded (Grassmann) kernel

Pairing $\bigwedge^{\star} X^{\mathscr{K}\,*} \otimes \bigwedge^{\star} X^{\mathscr{K}} \longrightarrow \mathbb{R}$, for all $\boldsymbol{\sigma}_k > \Longrightarrow (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k) \mid \boldsymbol{\sigma}_1 > \cdots > \boldsymbol{\sigma}_k$,

$$\big\langle \varphi(x^*), \psi(x) \big\rangle \stackrel{\text{def}}{=\!=} \varphi_0\, \psi_0 + \sum_{k=1}^{n} \varphi_k \psi_k + \sum_{k=1}^{n} \sum_{\boldsymbol{\sigma}_k <} \varphi_{\boldsymbol{\sigma}_k \cdots \boldsymbol{\sigma}_1}\, \psi_{\boldsymbol{\sigma}_1 \cdots \boldsymbol{\sigma}_k}$$

$$= |\psi_0|^2 + \sum_{k=1}^{n} \int_{\boldsymbol{\sigma}_k <} |\psi_{\boldsymbol{\sigma}_1 \cdots \boldsymbol{\sigma}_k}|^2\, d^n x, \quad \forall\, |\psi|^2 \propto |\varphi|^2 \in \mathbb{R},\ \frac{n}{2} \in \mathbb{Z}$$

such that for the dual space, graded basis $x^*_{\boldsymbol{\sigma}_k >}$,

$$\bigwedge^{\star} X^{\mathscr{K}} \ni \psi(x) = \psi_0 + \sum_{k=1}^{n} \sum_{\boldsymbol{\sigma}_k <} \psi_{\boldsymbol{\sigma}_k <}\, x_{\boldsymbol{\sigma}_k <} \quad \Big|\ \bigwedge^{k} X^{\mathscr{K}} \ni \sum_{\boldsymbol{\sigma}_k <} \psi_{\boldsymbol{\sigma}_k <}\, x_{\boldsymbol{\sigma}_k <}$$

$$\bigwedge^{\star} X^{\mathscr{K}\,*} \ni \varphi(x^*) = \varphi_0 + \sum_{k=1}^{n} \sum_{\boldsymbol{\sigma}_k >} \varphi_{\boldsymbol{\sigma}_k >}\, x^*_{\boldsymbol{\sigma}_k >} \quad \Big|\ \bigwedge^{k} X^{\mathscr{K}\,*} \ni \sum_{\boldsymbol{\sigma}_k >} \varphi_{\boldsymbol{\sigma}_k >}\, x^*_{\boldsymbol{\sigma}_k >}$$

where $\bigwedge^{\star} X^{\mathscr{K}\,*}$ is the dual graded algebra to $\bigwedge^{\star} X^{\mathscr{K}}$ generated by

$$\left\{ \begin{matrix} x_0 = 1;\ x_{\boldsymbol{\sigma}_k <} = x_{\boldsymbol{\sigma}_1} \otimes \cdots \otimes x_{\boldsymbol{\sigma}_k} \\ \forall\, \boldsymbol{\sigma}_k \in \{1, \ldots, n\};\ \ k = 1, \ldots, n \end{matrix} \ \middle|\ \begin{matrix} x_{\boldsymbol{\sigma}_\xi} \otimes x_{\boldsymbol{\sigma}_\eta} + x_{\boldsymbol{\sigma}_\eta} \otimes x_{\boldsymbol{\sigma}_\xi} = 0 \\ \boldsymbol{\sigma}_k < \Longrightarrow (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k) \mid \boldsymbol{\sigma}_1 < \cdots < \boldsymbol{\sigma}_k \end{matrix} \right\}.$$

Fixing integrals on $\bigwedge^{\star} X^{\mathscr{K}}$, $\bigwedge^{\star} X^{\mathscr{K}*}$, $\bigwedge^{\star}(X^{\mathscr{K}*} \otimes X^{\mathscr{K}})$ by choosing

$$x_1, \ldots, x_n \in \bigwedge^n X^{\mathscr{K}}, \quad x_n^*, \ldots, x_1^* \in \bigwedge^n X^{\mathscr{K}*}$$

and

$$x_n^*, \ldots, x_1^*, \, x_1, \ldots, x_n \in \bigwedge^n X^{\mathscr{K}*} \otimes \bigwedge^n X^{\mathscr{K}}$$

then

$$\int \bigotimes_{\xi=1}^{k} x^*_{\sigma_\xi} \bigotimes_{\xi=1}^{k} x_{\tau_\xi} \, dx^* \, dx = \begin{cases} 0 & , \quad k \neq n \\ (-1)^{\left(t(\sigma) + t(\tau) + \frac{n(n-1)}{2}\right)} & , \quad k = n \end{cases}$$

$$t(\sigma) := (\sigma_1, \ldots, \sigma_n) \longrightarrow (1, \ldots, n)$$
$$t(\tau) := (\tau_1, \ldots, \tau_n) \longrightarrow (1, \ldots, n).$$

Lemma 3.1.

$$\left\langle \varphi(x^*), \psi(x) \right\rangle = \int \exp\Bigl(\sum_{\xi} x^*_\xi x_\xi\Bigr) \, \varphi(x^*) \, \psi(x) \, dx^* \, dx \, .$$

Proof. $\heartsuit$.

Lemma 3.2. *Let* $Y^{\mathcal{K}} : X^{\mathcal{K}} \longrightarrow X^{\mathcal{K}}$ *by*

$$\begin{aligned} \psi_{Y^{\mathcal{K}}}(x) &= \sum_{\{\xi\}_<\,,\,\{\eta\}_<} x_{\{\xi\}_<} Y_{\{\xi\}_<\{\eta\}_<} \psi_{\{\eta\}_<} \\ &= \psi_0 \oplus Y\,\psi_1 \oplus Y^{\otimes 2}\,\psi_2 \oplus \cdots \end{aligned}$$

then

$$\psi_{Y^{\mathcal{K}}}(w) = \\ = \int \exp\!\big(-x^* Y^{\mathcal{K}}\, w\big)\ \exp(-x^* x)\ \psi(x)\, dx^*\, dx.$$

Proof. $\heartsuit$.

Lemma 3.3.

$$\int \exp\!\big(-x^*\, Y^{\mathcal{K}}\, w\big)\ \exp(-x^* x)\ \exp\!\Big(-{W^{\mathcal{K}}}^{*} W^{\mathcal{K}}\, x\Big)\ dx^*\, dx \\ = \exp\!\big(-w^*\, W^{\mathcal{K}} X^{\mathcal{K}}\, w\big).$$

Proof. $\heartsuit$.

Remark 3.1. Thus, $\exp(-w^* Y^{\mathcal{K}} w)$ is $Y^{\mathcal{K}}$ "integral kernel" acting on $\bigwedge^n X^{\mathcal{K}}$.

3.2 Vertex operators

(i). The Fermionic Fock space F i.e. $\left\langle X_m^{\mathscr{K}} \right\rangle \in \mathbb{C}^{\mathbb{Z}+\frac{1}{2}}$ is given by

$$F = \left\{ X_{m_1}^{\mathscr{K}} \wedge X_{m_2}^{\mathscr{K}} \wedge \cdots \;\middle|\; \begin{array}{l} m_\xi \in \mathbb{Z}+\frac{1}{2} \\ m_{\xi+1} = m_\xi - 1 \\ \xi \gg 1 \end{array} \right\}.$$

(ii). The Clifford algebra is given by

$$Cl_{\mathbb{Z}} = \left\langle \psi_m \, , \, \psi_m^* \right\rangle \;\; \left| \begin{array}{l} m \in \mathbb{Z} + \frac{1}{2} \\ \psi_m \psi_{m'} + \psi_{m'} \psi_m \; = \; \psi_m^* \psi_{m'}^* + \psi_{m'}^* \psi_m^* \; = \; 0 \\ \psi_m \psi_{m'}^* + \psi_{m'}^* \psi_m \; = \; \delta_{m\,m'} \, . \end{array} \right.$$

(iii). The Clifford algebra acting on the Fock space F:

$$\begin{aligned} \psi_m \, x_{m_1} \wedge x_{m_2} \wedge \cdots \;\; &= \;\; x_m \wedge x_{m_1} \wedge x_{m_2} \wedge \cdots \\ \psi_m^* \, x_{m_1} \wedge x_{m_2} \wedge \cdots \;\; &= \;\; \sum_{\xi=1}^{\infty} (-1)^{\xi} \; \delta_{m_\xi, m} \; x_{m_1} \wedge \cdots \wedge \widehat{x_{m_1}} \wedge \cdots \end{aligned}$$

(iv). The Heisenberg algebra is given by

$$\left\langle \alpha_n \right\rangle \;\left|\; \begin{array}{l} n \in \mathbb{Z}\backslash\{0\} \\ [\alpha_n,\, \alpha_{n'}] \;=\; -n\,\delta_{n,\,-n'}\,. \end{array}\right.$$

(v). The Heisenberg algebra acting on the Fock space F:

- As part of Bose-Fermi correspondence in 1D:

$$\alpha_n \longmapsto \sum_{m\,\in\,\mathbb{Z}+\frac{1}{2}} \psi_{m+n}\, \psi_m^*\,.$$

- As operator in F:

$$\Big[\alpha_n\,,\,\psi_\xi\Big] = \psi_{\xi+n}\;,\quad \Big[\alpha_n\,,\,\psi_\xi^*\Big] = -\psi_{\xi-n}^*\,.$$

(vi). The vertex operators in F are given by

$$X_\pm^{\mathcal{K}}(x) = \exp\Big(\sum_{n=1}^{\infty} \frac{x^n}{n}\,\alpha_{\pm n}\Big) \;\left|\; \begin{array}{l} \big(X_-^{\mathcal{K}}(x)v,\, w\big) = \\ \qquad = \big(v,\, X_+^{\mathcal{K}}(x)w\big) \\ \qquad = \big(X_+^{\mathcal{K}}(x)w,\, v\big). \end{array}\right.$$

(vii). The commutation relations are given by

$$
\begin{aligned}
X_+^{\mathcal{K}}(x)\, X_-^{\mathcal{K}}(y) &= (1-x) \cdot X_-^{\mathcal{K}}(y)\, X_+^{\mathcal{K}}(x) \\
X_+^{\mathcal{K}}(x)\, \psi(z) &= (1-z^{-1}x)^{-1} \cdot \psi(z)\, X_+^{\mathcal{K}}(x) \\
X_-^{\mathcal{K}}(x)\, \psi(z) &= (1-x\,z)^{-1} \cdot \psi(z)\, X_-^{\mathcal{K}}(x) \\
X_+^{\mathcal{K}}(x)\, \psi^*(z) &= (1-z^{-1}x) \cdot \psi^*(z)\, X_+^{\mathcal{K}}(x) \\
X_-^{\mathcal{K}}(x)\, \psi^*(z) &= (1-z\,x) \cdot \psi^*(z)\, X_-^{\mathcal{K}}(x).
\end{aligned}
$$

(viii). The eigenvectors are given by

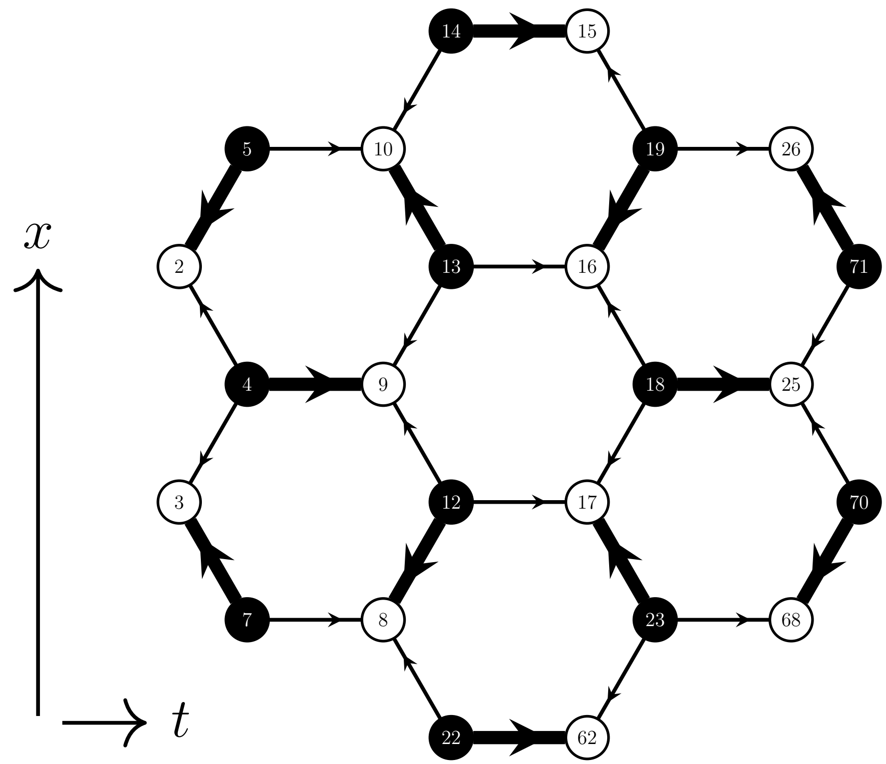
$$
X_-^{\mathcal{K}}(x) \prod_{\xi} \psi^*(w_\xi) \prod_{\eta} \psi^*(z_\eta)\, v_0^{(n)} =
$$

$$
= \prod_{\xi} (1-x\,z_\xi)^{-1} \prod_{\eta} (1-x\,w_\eta) \prod_{\xi} \psi^*(w_\xi) \prod_{\eta} \psi^*(z_\eta)\, v_0^{(n)}
$$

where $v_0^{(n)} = v_{n-\frac{1}{2}} \wedge v_{n-\frac{3}{2}} \wedge \cdots$

3.3 $\mathscr{K}$ Fermionic operators

For the one cube X^* of two-color tiles on bipartite hexagonal lattice X:



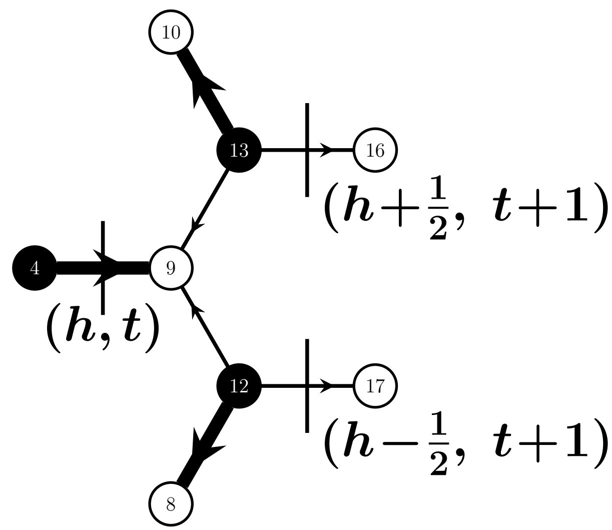
let the general parameterization for bipartite hexagonal lattice be given by

$$b(h,t) = (h,\ t - \tfrac{1}{2}),$$
$$w(h,t) = (h,\ t + \tfrac{1}{2}).$$

By above-given $b \sim w$ lattice, then the $(X^{\mathscr{K}}_{\xi\eta})$-inverse i.e. for observable:

$$K(h,t) \;=\; (h,t) \;-\; \big(h+\tfrac{1}{2},\, t+1\big) \;+\; y_{h,t}\,\big(h-\tfrac{1}{2},\, t+1\big).$$

Placing Fermions $x^*_{h,t}$, $x_{h,t}$ respectively at $b(h,t)$ and $w(h,t)$:

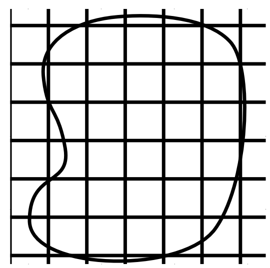


then

$$x^*Kx \;=\; \sum_{h,t} x^*_{h,t}\, x_{h,t} \;-\; \sum_{h,t} x^*_{h+\frac{1}{2},t+1}\, x_{h,t} \;+\; \sum_{h,t} x^*_{h-\frac{1}{2},t+1}\, x_{h,t}\; y_{h,t}$$

$$=\; \sum_{t} \big(x^*_t\, x_t \;+\; x_t\, V\, x^*_{t+1} \;+\; x_t\, V^{-1}\, x_t\; x^*_{t+1}\big).$$

Theorem 3.1. *Assuming $x_{h,t}\!=\!x_t$, analogous to notation $q_{h,t}\!=\!q_t$, then*



$$\Bigg|\ \textit{Prob}(\pi) \propto \prod_t q_t^{|\pi(t)|}$$

where the boundary conditions imply that

$$\mathcal{Z} = \int \exp\big(x^* Y^{\mathscr{K}}\, x\big)\, dx^*\, dx \ =$$

$$= \Big\langle X_-^{\mathscr{K}}\big(x_{-\frac{1}{2}}\big) \cdots X_-^{\mathscr{K}}\big(x_{u_0+\frac{1}{2}}\big)\, X_+^{\mathscr{K}}\big(x_{\frac{1}{2}}\big) \cdots X_+^{\mathscr{K}}\big(x_{u_1+\frac{1}{2}}\big)\, v_0^{(0)}\,,\ \ v_0^{(0)} \Big\rangle\,.$$

Proof (outline).

$$\int \cdots \exp\big(x^*_{t-1}\, x_{t-1}\big) \cdot \exp\big(x_{t-1}\, (V - V^{-1} X^{\mathscr{K}}_t)\, x^*_t\big) \cdot$$
$$\cdot \exp\big(x^*_t\, x_t\big) \cdot \exp\big(x_t\, (V - V^{-1} X^{\mathscr{K}}_t)\, x^*_{t+1}\big) \cdots$$

$$= \cdots \underbrace{\widetilde{\big(V - V^{-1} X^{\mathscr{K}}_{t-1}\big)}^{-1}}_{X^{\mathscr{K}}_+(x_t)} \cdot \underbrace{\widetilde{\big(V - V^{-1} X^{\mathscr{K}}_t\big)}^{-1}}_{X^{\mathscr{K}}_-(x_t)} \cdots$$

where $X^{\mathscr{K}}_+(x_t)$ and $X^{\mathscr{K}}_-(x_t)$ each depends on t such that

$$\widetilde{Y^{\mathscr{K}}} = Y^{\mathscr{K}}, \text{ where } V \hookleftarrow \text{ is lifted to } \textstyle\bigwedge^{\frac{\infty}{2}} V \quad \Big|\, V = \bigoplus_{h \in \mathbb{Z}+\frac{1}{2}} \mathbb{C}\, v_h$$

under boundary conditions, etc. □

Remark 3.2. Direct proof exists combinatorially besides the $\mathscr{K}$ way.

Corollary 3.1.

$$\mathcal{Z} = \prod_{m=\frac{1}{2}}^{u_1 - \frac{1}{2}} \prod_{m' = u_0 + \frac{1}{2}}^{-\frac{1}{2}} \big(1 - x^-_{m'}\, x^+_m\big)^{-1}.$$

Theorem 3.2 (Okounkov & R., 2005). *Following* $(X^{\mathcal{K}}_{\xi\eta})$*-inverse then*

$$\left\langle \sigma_{(h_1 t_1)} \cdots \sigma_{(h_k t_k)} \right\rangle = \det(K((t_\xi, h_\xi), (t_\eta, h_\eta)))_{1 \leqslant \xi, \eta \leqslant k}$$

$$K((t_\xi, h_\xi), (t_\eta, h_\eta)) =$$

$$= \frac{1}{(2\pi i)^2} \int\limits_{|z| < R(t_1)} \int\limits_{|z| < \widetilde{R}(t_2)} \frac{\Phi_-(z, t_1) \ \ \Phi_+(w, t_2)}{\Phi_+(z, t_1) \ \ \Phi_-(w, t_2)} \cdot$$

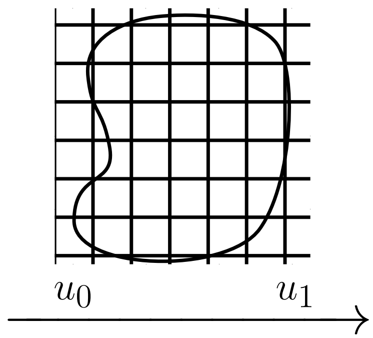
$$\cdot \frac{1}{z - w} \cdot z^{\left(-h_1 - B(t_1) - \frac{1}{2}\right)} \cdot w^{\left(h_2 - B(t_2) - \frac{1}{2}\right)} \, dz \, dw$$

where

$$\begin{array}{l|l} |w| < |z|, \ t_1 \geqslant t_2 & R(t) = \min\limits_{m > t}((x_m^+)^{-1}), \ \widetilde{R}(t) = \max\limits_{m < t}(x_m^-), \ B(t) = \frac{|t|}{2} - \frac{|t - u_0|}{2} \\ |w| > |z|, \ t_1 < t_2 & \Phi_+(z, t) = \prod\limits_{m > \max(t, \frac{1}{2})} (1 - z \, x_m^+), \ \ \Phi_-(z, t) = \prod\limits_{m < \max(t, -\frac{1}{2})} (1 - z^{-1} \, x_m^-). \end{array}$$

Proof. $\heartsuit$.

3.4 Thermodynamic limit with scaling



$$\left.\begin{aligned} x_m^+ &= a q^m \\ x_m^- &= a^{-1} q^m \end{aligned}\right\} \text{assumed}$$

corresponding to $\text{Prob}(\pi) \propto q^{\,|\pi|}$

Consider the limit $q = e^{-\varepsilon}$, $\varepsilon \longrightarrow 0$, $u_1 = \frac{1}{\varepsilon} v_1$, $u_0 = \frac{1}{\varepsilon} v_0$; fixed v_1, v_0:

$$\mathcal{Z} = \prod_{\substack{u_0 < n < 0 \\ 0 < m < u_1}} (1 - x_m^- x_n^+)^{-1} = \prod_{\substack{u_0 < n < 0 \\ 0 < m < u_1}} (1 - q^{m-n})^{-1}$$

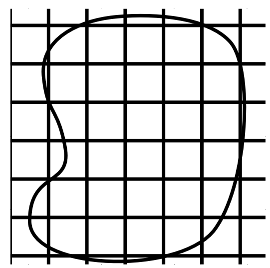
$$\langle\, |\pi|\,\rangle = q \frac{\partial}{\partial q} \ln \mathcal{Z} = \varepsilon^{-3} \int_0^{u_1} \int_{u_0}^{0} \underbrace{\frac{s-t}{1-e^{t-s}}}_{\text{3D volume}} \, ds\, dt + \cdots$$

where

$$\ln \mathcal{Z} = \varepsilon^{-2} \int_0^{u_1} \int_{u_0}^{0} \ln(\underbrace{1-e^{-s+t}}_{\substack{\text{2D partition}\\ \text{function}}}) \, ds\, dt + \cdots$$

3.5 Asymptotics of correlation function

Consider $\varepsilon \longrightarrow 0$, for $t_\xi = \varepsilon^{-1}\tau_\xi$, $h_1 = \varepsilon^{-1}\chi_\xi$, with fixed τ_ξ, χ_ξ:



$(\tau_\xi,\ \chi_\xi)$
in the bulk

then in evaluation by steepest decent

$$K((t_1,h_1),(t_2,h_2)) \longrightarrow$$

$$\longrightarrow \frac{1}{(2\pi i)^2} \int_{C_z} \int_{C_w} \exp\Big(\varepsilon^{-1}(S(z,t_1,\chi_1) - S(w,t_2,\chi_2))\Big) \cdot \cdot\ (zw)^{1/2}\ (z-w)^{-1}\ dz\ dw$$

for

$$S(z,t,\chi) = = -(\chi + \frac{\tau}{2} - u_0)\ \ln \mathcal{Z} + \mathsf{Li}_2(ze^{-v_0}) + \mathsf{Li}_2(ze^{-v_1}) - \mathsf{Li}_2(z) - \mathsf{Li}_2(ze^{-\tau})$$

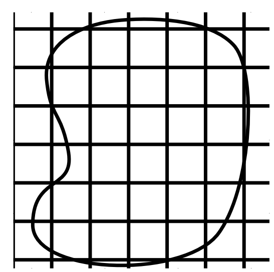
$$\mathsf{Li}_2(z) = = \int_0^z t^{-1}\ \ln(1-t)\ dt.$$

3.6 Critical point discriminants

The equality

$$\exp\left(\chi+\frac{\tau}{2}\right) \;=\; \frac{(1-ze^{-v_0})(1-ze^{-v_1})}{(1-z)(1-ze^{-\tau})}$$

gives quadratic equation, that is, a discriminant for two real, or two complex conjugate solutions, or the discriminant is zero, respectively, on the outside, within, or on the boundary given by a space of curves



moreover,

$$\partial_\chi h_0(\tau,\chi) \;=\; \tfrac{1}{\pi}\,\mathrm{arg}(z_0)$$

$$\left\langle \sigma_{(h,\,t)}\right\rangle = K((t,h),\,(t,h)) \longrightarrow \varepsilon\,\partial_\chi h_0(\tau,\chi).$$

3.7 Asymptotics steepest descent

$$K((t_1,h_1),(t_2,h_2)) = -\frac{\varepsilon}{2\pi}\cdot\Big(\frac{\exp\{\varepsilon^{-1}(S_1(z_1)-S_2(w_2))\}}{(z_1-w_2)\sqrt{-w_2S_2''(w_2)}\;\sqrt{z_1S_1''(z_1)}} -$$

$$-\frac{\exp\{\varepsilon^{-1}(S_1(z_1)-S_2(\overline{w}_2))\}}{(z_1-\overline{w}_2)\sqrt{-\overline{w}_2S_2''(\overline{w}_2)}\;\sqrt{z_1S_1''(z_1)}} + c.\,c.\Big)\cdot\big(1+\mathcal{O}(1)\big)$$

That is, for $\mathcal{H}_+ = \{z\in\mathbb{C},\ \mathrm{Im}\, z>0\} \mid z_0(\chi,\tau) =$ inner process, such that

$$z_1 = z_0(\chi_1,\tau_1)$$

$$w_2 = z_0(\chi,\tau)$$

$$K((t_1,h_1),(t_2,h_2)) =$$

$$= \frac{\varepsilon}{2\pi}\exp\{\varepsilon^{-1}(\mathrm{Re}(S(z_0(\chi_1,\tau_1)))-\mathrm{Re}(S(z_0(\chi_2,\tau_2))))\}\cdot$$

$$\cdot\Big(\frac{\exp\{i\,\varepsilon^{-1}(\mathrm{Im}(S'(z_1))-\mathrm{Im}(S(w_2)))\}}{(z_1-w_2)} +$$

$$+\frac{\exp\{i\,\varepsilon^{-1}(\mathrm{Im}(S'(z_1))-\mathrm{Im}(S(\overline{w}_2)))\}}{(z_1-\overline{w}_2)} + c.\,c.\Big)\cdot\big(1+\mathcal{O}(1)\big) \qquad (*).$$

Remark 3.3. Implies convergence of $\mathcal{K}$-Fermions to free Dirac-Fermions:

$$\frac{1}{\sqrt{\varepsilon}}\psi_{\vec{x}} = \exp\{\varepsilon^{-1}\mathsf{Re}(S(z_0))\}\cdot\Big(\psi_+(z_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0))) + \\ + \psi_-(\overline{z}_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0)))\Big)\cdot\big(1+\mathcal{O}(1)\big)$$

$$\frac{1}{\sqrt{\varepsilon}}\psi^*_{\vec{x}} = \exp\{\varepsilon^{-1}\mathsf{Re}(S(z_0))\}\cdot\Big(\psi^*_+(z_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0))) + \\ + \psi^*_-(\overline{z}_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0)))\Big)\cdot\big(1+\mathcal{O}(1)\big)$$

where

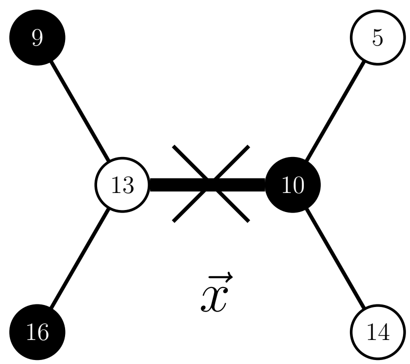
$$\mathbb{E}(\psi^*_\pm(z)\,\psi_\pm(w)) = \frac{1}{z-w}$$

$$\mathbb{E}(\psi^*_\pm(z)\,\psi_\mp(w)) = \mathbb{E}(\psi^*\,\psi^*) = \mathbb{E}(\psi\,\psi) = 0$$

such that $\psi^*_\pm(z)$, $\psi_\pm(w)$ are spinors:

$$\psi^*_\pm(z) = \psi^*_\pm(w)\sqrt{\frac{\partial w}{\partial z}}\,, \quad \psi_\pm(z) = \psi_\pm(w)\sqrt{\frac{\partial w}{\partial z}}\,.$$

The observable is given by:



$$\Big\langle (\boldsymbol{\sigma}_{\vec{x}_1} - \langle \boldsymbol{\sigma}_{\vec{x}_1}\rangle)\ (\boldsymbol{\sigma}_{\vec{x}_2} - \langle \boldsymbol{\sigma}_{\vec{x}_2}\rangle)\Big\rangle \ = \ K_{12}K_{21} \ =$$

$$= \frac{\varepsilon^2}{(2\pi)^2}\left(\frac{\frac{\partial z_1}{\partial x_1}\frac{\partial w_2}{\partial x_2}}{(z_1 - w_2)^2} \ - \ \frac{\frac{\partial z_1}{\partial x_1}\frac{\partial \overline{w}_2}{\partial x_2}}{(z_1 - \overline{w}_2)^2} \ + \ c.\,c.\right) \times$$

$$\times \ \big(1 + \mathcal{O}(1)\big).$$

In particular,

$$\boldsymbol{\sigma}_{\vec{x}_1} - \langle \boldsymbol{\sigma}_{\vec{x}_1}\rangle = \ \varepsilon\, \partial_x\, \varphi(z_0(\tau, x)) \ + \ \cdots \quad \Big|\ \varphi(z) = \text{Gaussian free field on } \mathcal{H}_+$$

such that the Green's function of Dirichlet problem on $\mathcal{H}_+$ is given by

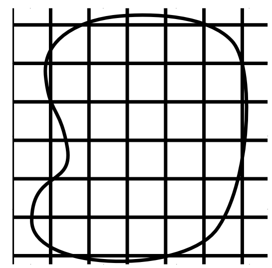
$$\langle \varphi(z)\, \varphi(w)\rangle = \ \frac{1}{2\pi}\, \ln\left|\frac{z - w}{z - \overline{w}}\right|$$

and, the Bose-Fermi correspondence is given by

$$\partial_x \varphi \ = \ : \widetilde{\psi}(z, \overline{z}) \ \ \widetilde{\psi}(z, \overline{z}) : \ \cdots \ .$$

3.8 Scaling limit within the $\mathscr{K}$ operator

Let $X = D_\varepsilon = \varphi_\varepsilon(L) \cap D$; $A_X^{\mathscr{K}}$ = difference operator, for arbitrary lattice L:



where $\varepsilon \longrightarrow 0$ asymptotics is locally allowed for some $\mathscr{G}_{x,y}$ equation

$$(A_X^{\mathscr{K}})_x \cdot \mathscr{G}_{x,y} \;=\; \delta_{x,y}\,.$$

Case 3.1.
(i) Hexagonal lattice: Uses the weighted as above, for

$$q_t = e^{-\varepsilon\, f(t)}, \quad t = \frac{\tau}{\varepsilon}, \quad \varepsilon \longrightarrow 0.$$

Theorem 3.3. $\mathscr{G}_{x,y} =$ *same as* $(*)$, *with different* $z_0(\tau, x)$.

Proof. $\heartsuit$.

(ii) Periodic lattice: Utilizes variational principle.

3.9 Limit shapes and the variational principle

(i). For the $N \times M$ torus

$$\mathcal{Z}(H,V) = \sum_{D} \prod_{k \cap D} \omega_k \times \exp(H\Delta_a h_D + V\Delta_b h_D)$$

$$= \frac{1}{2}\left\{\mathrm{Pf}\left(A^{K_1}\right) + \mathrm{Pf}\left(A^{K_2}\right) + \mathrm{Pf}\left(A^{K_3}\right) - \mathrm{Pf}\left(A^{K_4}\right)\right\}$$

where $N, M \longrightarrow \infty$, for fixed $\dfrac{N}{M}$.

Let $\omega_k = 1$, then using Fourier transform the $X^{\mathcal{K}}$ eigenvalues can be found.

Theorem 3.4 (McCoy & Wu, 1969; Kenyon & Okounkov, 2005).

$$\lim_{N,M\longrightarrow\infty} \frac{1}{NM} \ln \mathcal{Z}_{NM} = \oint\oint \ln|1+zw| \; \frac{dz}{z}\,\frac{dw}{w}$$

$$= f(H,V) \; \Bigg| \; \begin{array}{l} |z| = e^H \\ |w| = e^V. \end{array}$$

(ii). Taking Legendre transform

$$\boldsymbol{\sigma}(s,t) \;=\; \max_{H,V}\,(H_s + V_t - f(H,V))$$

then

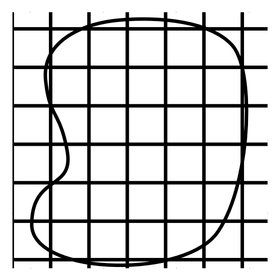
$$\sum_{D} 1 \;=\; \sum_{D}\;\prod_{k\cap D} \boldsymbol{\omega}_k \;=\; \exp\left\{NM\,\boldsymbol{\sigma}(s,t)\cdot\big(1+\mathcal{O}(1)\big)\right\}$$

where

$$\frac{\Delta_a h_D}{N} = s, \quad \frac{\Delta_b h_D}{M} = t, \quad N, M \longrightarrow \infty, \quad \frac{N}{M} \text{ fixed.}$$

(iii). Take domain

$\Delta_a h = sN,\; \Delta_b h = tM.$



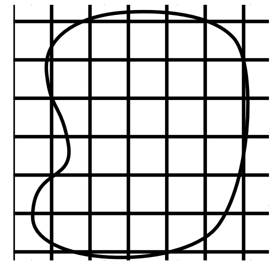
Theorem 3.5 (Cohn, Kenyon, & Propp, 2000).

$$\sum_{D} 1 \;=\; \exp\left\{NM\,\boldsymbol{\sigma}(s,t)\cdot\big(1+\mathcal{O}(1)\big)\right\}$$

with the boundary conditions of height function h_D.

(iv). Take domain

$\Delta_a h = sN,\ \Delta_b h = tM.$



$$\mathcal{Z}_{D_\varepsilon} = \sum_{\left\{\substack{\text{values of}\\ \text{height functions}\\ \text{on}\\ \text{boundaries}\\ \text{between rectangles}}\right\}} \mathcal{Z}_{\substack{\square N_\xi \\ M_\eta}}(h_{\text{bound}})$$

$$= \sum_{\{\Delta_x h,\ \Delta_y h\}_{\xi\eta}} \exp\left(\sum_{\substack{\square N_\xi \\ M_\eta}} N_\xi\, M_\eta\ \sigma\!\left(\frac{\Delta_x h}{M_\eta}, \frac{\Delta_y h}{N_\xi}\right)\right)$$

$$= \exp\left(\varepsilon^{-2}\int_D \sigma(\partial_x h_0,\ \partial_y h_0)\ dx\, dy\ \big(1 + \mathcal{O}(1)\big)\right)$$

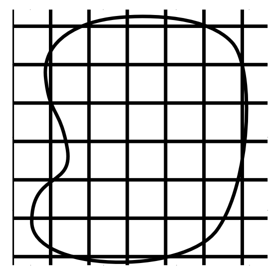
where $h_0 =$ minimizer for

$$S[h] = \int_D \sigma(\partial_x h_0,\ \partial_y h_0)\ dx\, dy.$$

Theorem 3.6 (Cohn, Kenyon, & Propp, 2000).

$$\lim_{\varepsilon \longrightarrow 0} \varepsilon^2 \ln \mathcal{Z}_{D_\varepsilon} = \int_D \sigma\big(\vec{\nabla} h_0\big)\, dx\, dy$$

on domain



such that:

$0 < \partial_x h, \partial_y h < 1$

$h_0 =$ *minimizer i.e. the limit shape*

$h_0\big|_{\partial D} = b,$ *the boundary condition which appears in the limit* $\varepsilon \longrightarrow 0$

$h = \frac{1}{\varepsilon} h_0 + \cdots ,$ *the height function*; $\varphi = \frac{1}{\varepsilon}(h_0 + \varepsilon\varphi),$ *the fluctuations.*

3.10 Physics way of the higher genus observable

$$S[h_0+\varepsilon\varphi] = S[h_0] + \frac{\varepsilon^2}{2}\iint_D a^{\xi\eta}(x)\partial_\xi\varphi\,\partial_\eta\varphi\,d^2x$$

$$a^{\xi\eta}(x) = \partial_\xi\partial_\eta\,\varphi(s,t)\Big|_{\substack{s=\partial_1 h_0\\ t=\partial_2 h_0}}$$

such that:

Partition function equals

$$\mathcal{Z} = \exp(\varepsilon^{-2}S(h_0))\int\exp\left(\frac{1}{2}\iint_D a^{\xi\eta}(x)\partial_\xi\varphi\,\partial_\eta\varphi\,d^2x\right)d\varphi$$

where $D =$ scalar field with Riemannian metric induced by h_0;

and, correlation equals

$$\langle\varphi(x)\,\varphi(y)\rangle = \mathcal{G}(x,y)$$

where $\mathcal{G} =$ Green's function for $\Delta = \partial_\xi(a^{\xi\eta}\,\partial_\eta)$.

Conjecture 3.1. $\mathcal{G}$ *is globally same as by* $\mathcal{K}$ *operator asymptotics* $(*)$.

Remark 3.4. The conjecture is theorem in certain cases as earlier-given.

(Chebotarev, Guskov, Ogarkov & Bernard, 2019). *For free-action or interaction quantum field theory (QFT)* $\mathcal{G}[g,\bar{\varphi}] = -\ln \mathcal{S}[g,\bar{\varphi}]$,

$$\mathcal{S}[g,\bar{\varphi}] \;=\; \lim_{N\to\infty} \mathcal{S}_N[g,\bar{\varphi}] \;=\; \frac{\mathcal{Z}[g, \eta = \hat{G}^{-1}\bar{\varphi}]}{\mathcal{Z}[\eta = \hat{G}^{-1}\bar{\varphi}]} \;=$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\{ \prod_{a=1}^{n} \int d\Gamma_a \, e^{i\lambda_a \, \bar{\varphi}(x_a)} \right\} e^{-\frac{1}{2} \sum\limits_{a,b=1}^{n} \lambda_a \, \lambda_b \, G(x_a - x_b)}$$

$$\mathcal{S}_N[g,\bar{\varphi}] = \int d\sigma_u e^{-\int d^D x g(x) U\left[\bar{\varphi}(x) + \sum\limits_s u_s \frac{D_s(x)}{\sqrt{1+q_s}}\right] - \frac{1}{2}\sum\limits_s \ln(1+q_s) + \frac{1}{2}\sum\limits_s \frac{q_s}{1+q_s} u_s^2}.$$

(Bernard, Guskov, Kalugin, Ivanov & Ogarkov, 2019). *In critical phase (non-)polynomial QFT*, *spatial measure* $d\mu$ *generating functional*,

$$\mathcal{Z}\,[g; d\mu] \;=\; \int d\sigma_t \prod\!\!\!\!\!\!\int^{d\mu(x)} \left\| e^{\,f[\varphi(x);\,x]} \right\|_1 \;=$$

$$= \mathcal{C}_1\,[g; d\mu] \left\{ 1 \,+\, \frac{\pi(1-\eta)}{\Gamma^2(\frac{1}{4})} \int \frac{d\mu(x)}{\sqrt{2g(x)}} \,+\, \mathcal{O}\left[\frac{1}{g}\right] \right\}.$$

Conclusion: the higher genus observable yet

1. How to make (simulate) such pictures of perfect-matching mixture by:

   (i). Monte Carlo for $\exp(\alpha\, 1000^2)$

   (ii). Sampling around most probable region by MCMC

2. How to describe such random surface invariant-limit analytically by:

   (i). Equipartition Pfaffian asymptotics with boundary conditions

   (ii). Variational principle: Minimizer functional in large deviation

References

[1] M. Bernard, V. Guskov, M. Ivanov, A. Kalugin, and S. Ogarkov. *Nonlocal scalar QFT: func. integration, basis func. rep. and strong coupl. expansion.* Particles, 2(3):385–410, 2019.

[2] I. Chebotarev, V. Guskov, S. Ogarkov, and M. Bernard. *S-matrix of nonlocal scalar QFT in basis func. rep.* Particles, 2(1):103–139, 2019.

[3] P. Kasteleyn. *Dimer statistics and phase transitions*. J. Math. Phys., 4:287–293, 1963.

[4] M. Kazarian and S. K. Lando. *Weight systems and invariants of graphs and embedded graphs*. Russ. Math. Surv., 77(5):893–942, 2022.

[5] S. Lando. *On primitive elements in the bialgebra of chord diagrams*. MPIM Preprint Series, 47:47, 1994.

[6] A. Okounkov and N. Reshetikhin. *Random skew plane partitions and the Pearcey process*. Comm. Math. Phys., 269:571–609, 2007.

Thank you!