Anomalous Fluctuating Wild Forest Bridges

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Abstract

We prove enveloping bialgebra $\mathcal{U}(\mathfrak{gl}_n^{\otimes})/\mathcal{O}(n)$, wild (random re-graft root-growth) forest bridge fluctuation in seemingly anomalous evolutionary loci of empirically known supertree genus order, Fourier-Stieltjes $\phi_k(t;\mu_\xi,\Sigma_{\xi\eta})$, 2-side Laplace $\theta_k(t;\mu_\xi,\Sigma_{\xi\eta})$ asymptotics Gauss-hypergeometric, Bessel, Euler-beta multivariate. CRISPR reducibility is lax on the continuum $2^{\aleph_\mathfrak{c}}$ (quarternion) algebra of supersolvable extremal theory.

Keyword: fixed-genus, wild-forest, topological-genus

Enveloping $\mathcal{U}(\mathfrak{gl}_n^{\otimes})/\mathcal{O}(n)$ invariant

On compact, metrizable randomized disk $[-1,+1]^\Omega \ni X=(X_t)_{t\geqslant 0}$ for $\overline{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$ distribution of (in)finite Markov subset-state: The time $t\geqslant 0$, chain follows confidence-level $1-\alpha$ and -interval $(\theta^*-\varepsilon,\ \theta^*+\varepsilon)$ on fixed genus g, ordinal $\mathfrak{c}$ continuum[1] optimum α^*, joint-weight hypothesis

$$\mathbb{P}[\theta^*-\varepsilon < X < \theta^*+\varepsilon] \ = \ 1-\alpha \quad \Big| \ X_{t_0}\in dx = 2\varepsilon,\ \ldots,\ X_{t_{n-1}}\in dx=2\varepsilon \tag{1}$$

for algebra $\mathcal{U}(\mathfrak{gl}_n^{\otimes})$ scheme θ^* of bounded decreasing confidence intervals on increasing significance level α, the type I error to reject a true null hypothesis of continuous $\mathbb{C}^n$ distribution $\Phi(x)$ for the random lattice $X_{[n]}=(X_{t_0},\ldots,X_{t_{n-1}})$, on symmetry $\mathcal{H}_0\colon\ 1-\Phi(x)-\Phi(-x)=0$; given

$$\varepsilon \geqslant \frac{\sqrt{N\log\log\log N}}{2} \tag{2}$$

for continuity in inverse $f(x)=d\Phi(x)$, transpose $\widetilde{y}$ of complex-conjugate $y\in\mathbb{C}^n$ with respect to X, i.e. $\operatorname{sgn}[\operatorname{Im}[-y]]=\operatorname{sgn}[\operatorname{Im}[X]]$, for multivariate Fourier-Stieltjes transform $\phi(y)=\mathbb{E}[e^{i\,\operatorname{Tr}[\operatorname{Re}[\widetilde{y}X]]}]$:

$$\int_\Omega e^{i\,\operatorname{Tr}\left[\operatorname{Re}\left[\widetilde{y}X\right]\right]} \underbrace{\underbrace{\frac{\partial}{\partial w_{t_0}}\cdots\frac{\partial}{\partial w_{t_{n-1}}}\Phi(x_{t_0},\ldots,x_{t_{n-1}})}_{f(x\in\mathbb{C}^n)}\,dx_{t_0}\cdots dx_{t_{n-1}}}_{\mathbb{P}(X\in dx)} \ \Bigg| \ \operatorname{Tr}[\operatorname{Re}[\widetilde{y}X]] = \operatorname{Re}\left[\sum_{\xi=0}^{n-1}\widetilde{y}_{t_\xi}X_{t_\xi}\right] \tag{3}$$

$$\begin{aligned}\Phi(x) \ &= \ \mathbf{P}\big(X_{t_0}\leqslant x_{t_0},\ldots,X_{t_{n-1}}\leqslant x_{t_{n-1}}\big)\\ &= \int_{-\infty}^{x_{t_0}}\cdots\int_{-\infty}^{x_{t_{n-1}}}\frac{\partial}{\partial w_{t_0}}\cdots\frac{\partial}{\partial w_{t_{n-1}}}\Phi(w_{t_0},\ldots,w_{t_{n-1}})\,dw_{t_0}\cdots dw_{t_{n-1}}.\end{aligned} \tag{4}$$

[1] where, in general, there exists no set S such that $|\aleph_\mathfrak{c}|<|S|<2^{|\aleph_\mathfrak{c}|}=|\aleph_{\mathfrak{c}+1}|$, for any infinite cardinal $|\aleph_\mathfrak{c}|$.

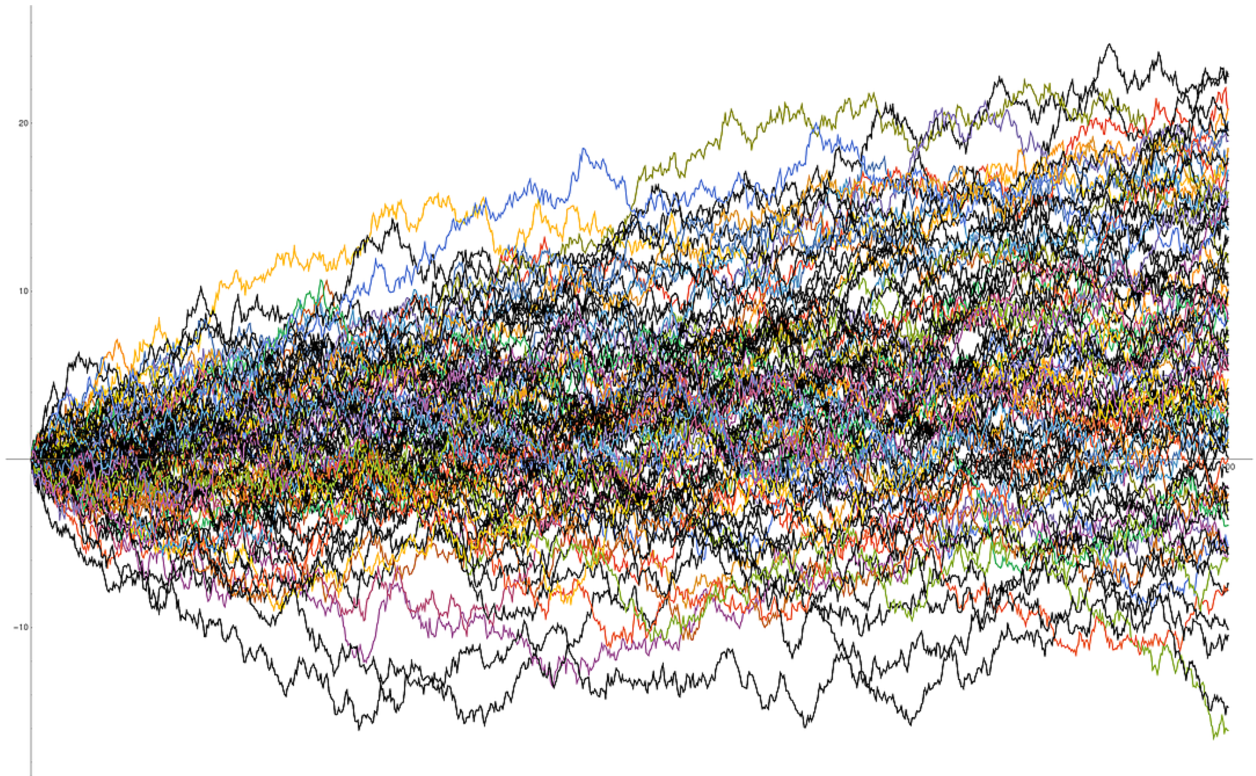

Fig. 1: 10^2 rays, Gaussian process of drift $\mu = 0.05$, risk $\sqrt{\Sigma} = 0.95$.

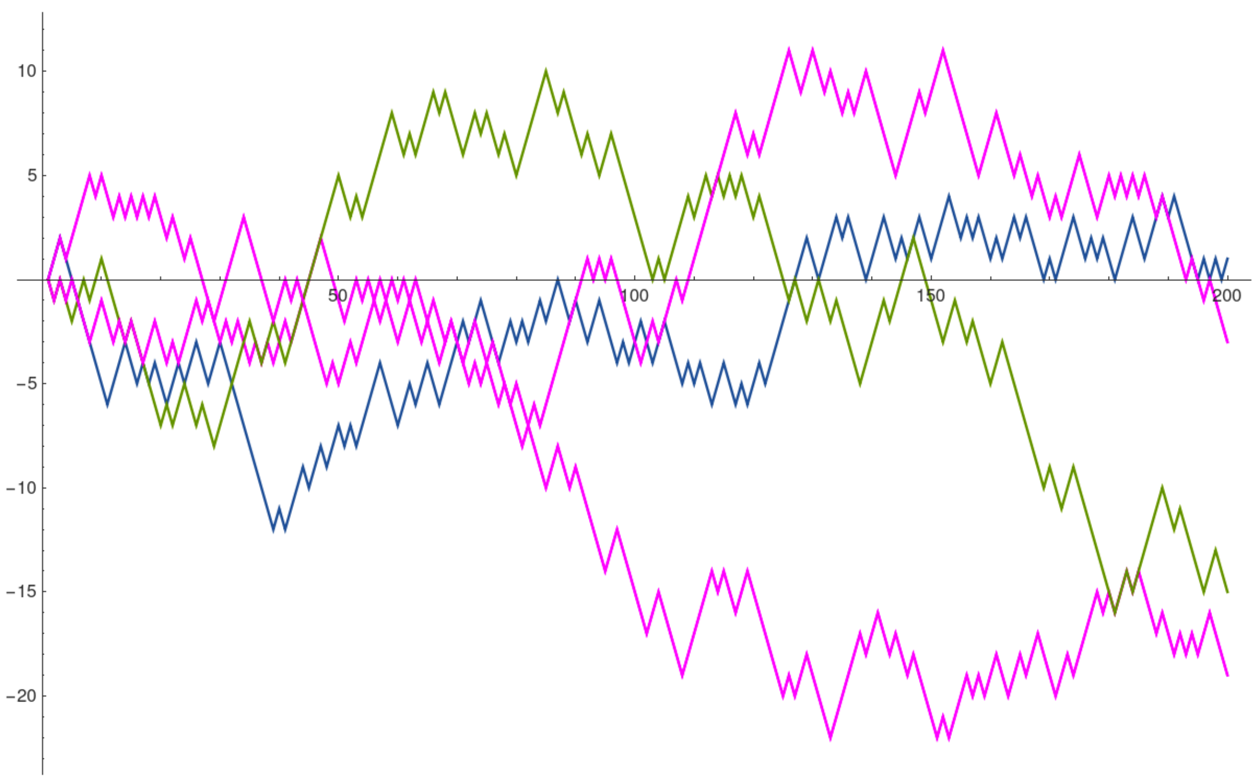

Fig. 2: point-source, 4 rays, symmetric Bernoulli $\{-1, +1\}$, for $\mu = 2^{1-k} - 1|_{k=1}$, $\Sigma = 2^{2-k} - 2^{2-2k}|_{k=1}$.

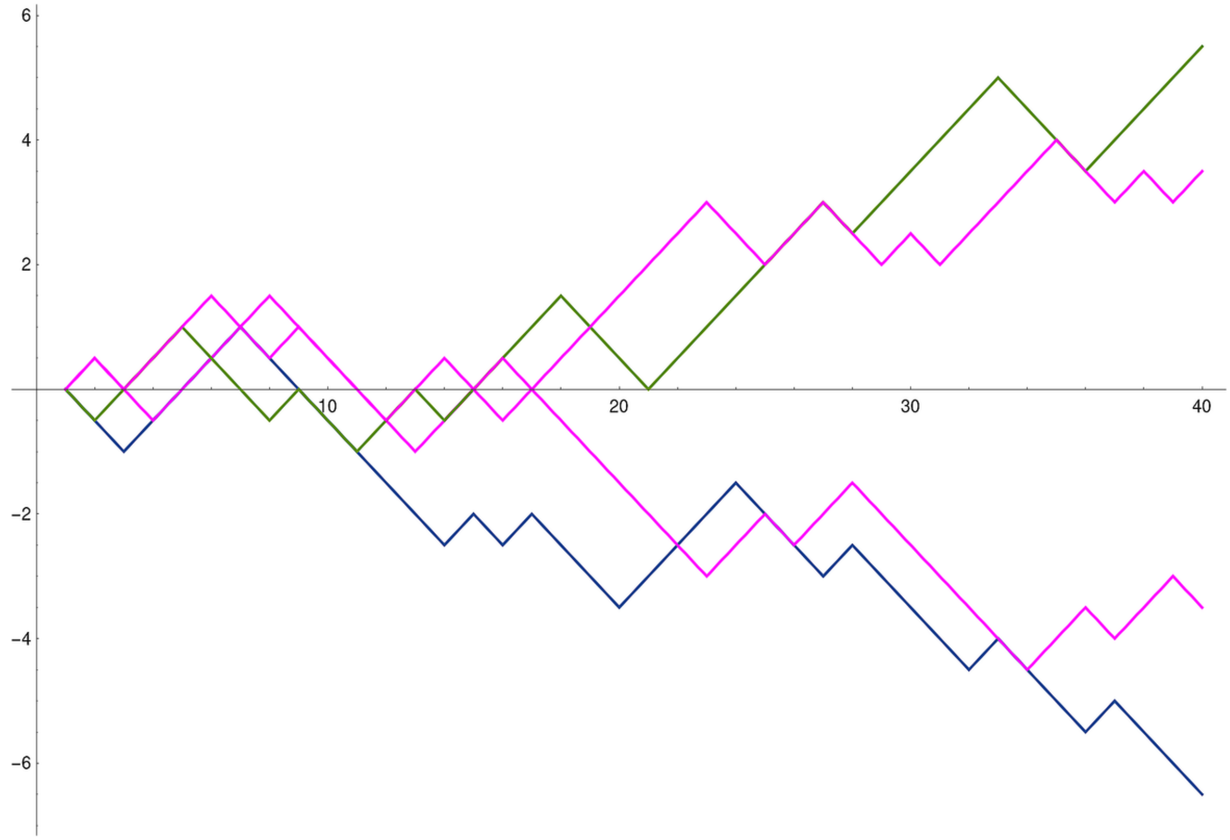

Fig. 3: point-source, 4 rays, symmetric Bernoulli $\{-1,+1\}$, for $\mu\!=\!2^{-k}\!-\!2^{-1}\big|_{k=1}$, $\Sigma\!=\!2^{-k}\!-\!2^{-2k}\big|_{k=1.}$

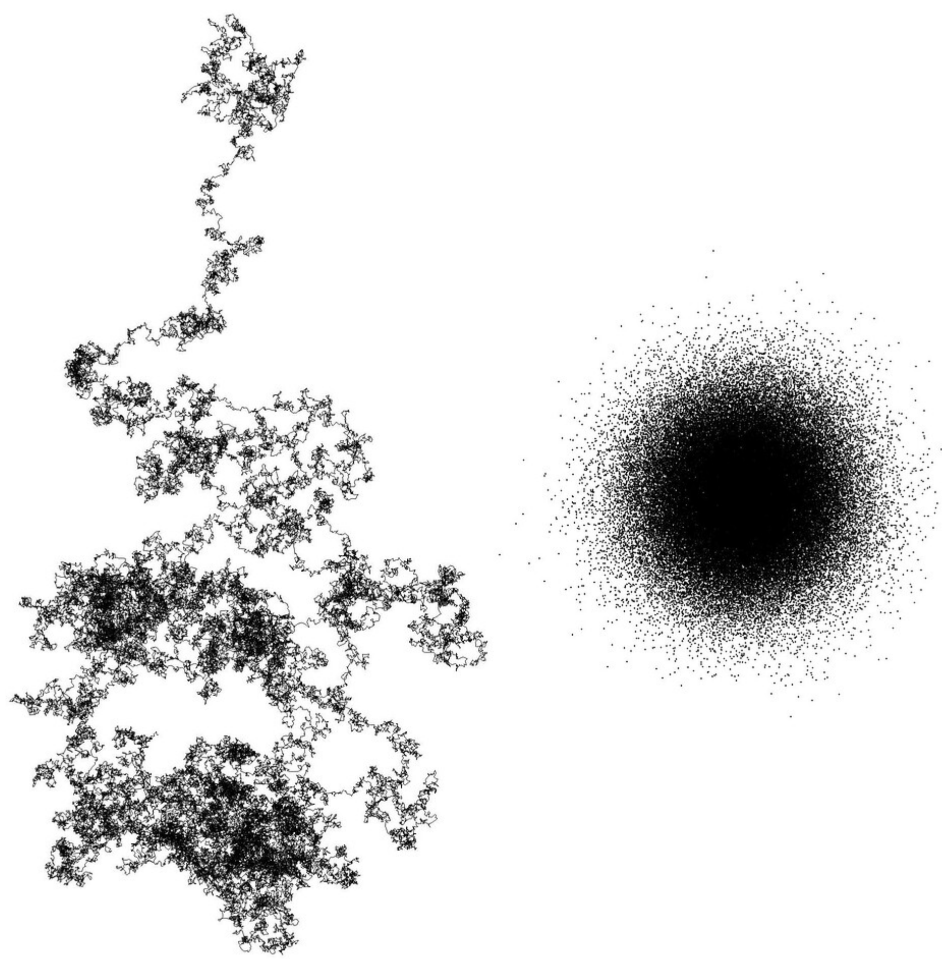

Fig. 4: 2-dimensional Gaussian walk and set of $\mathbb{E}[\text{distance}]\!=\!10^{5/2}$, for $\mu\!=\!(0)$, $\Sigma\!=\!((1,0),(0,1))$.

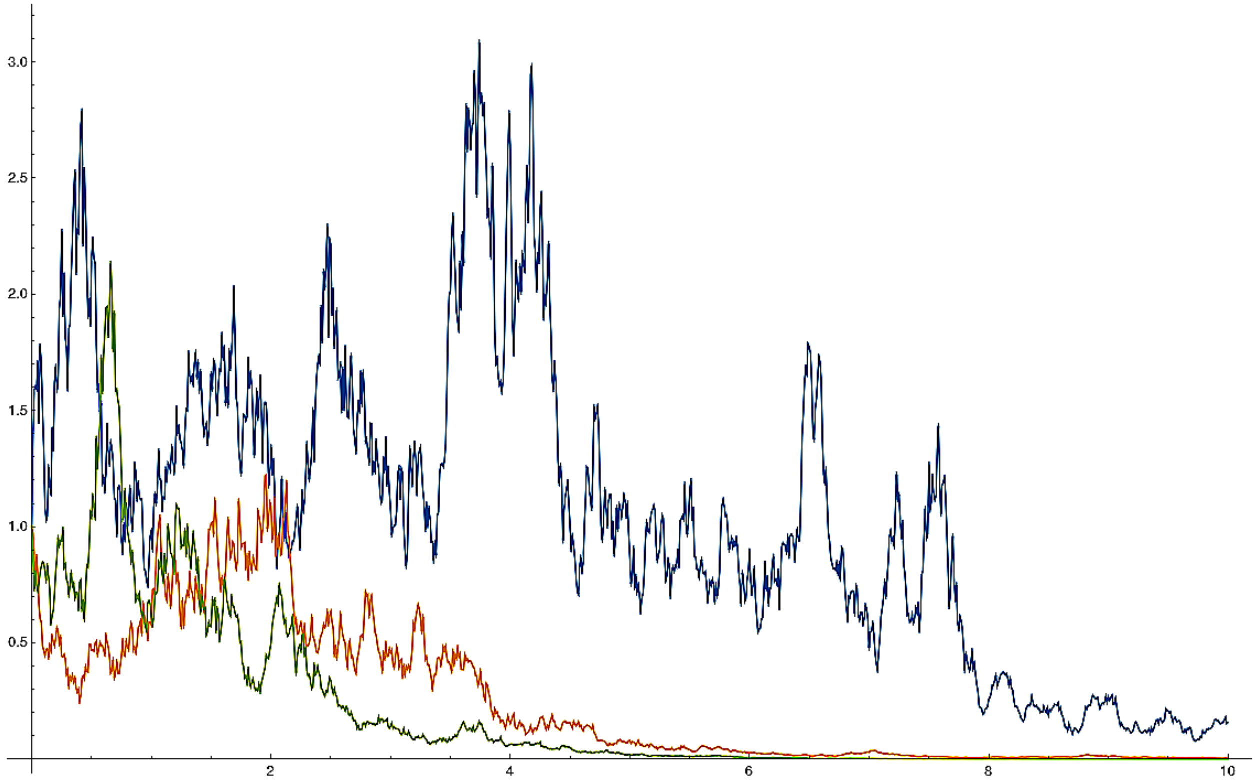

Fig. 5: point-source, Martingale process for $\mu = e^{t\nu + \frac{tW}{2} - \frac{t}{2}}$, $\Sigma = (e^{tW} - 1)\, e^{2(t\nu - \frac{t}{2}) + tW} \mid \nu = 0.05, \sqrt{W} = 0.95$.

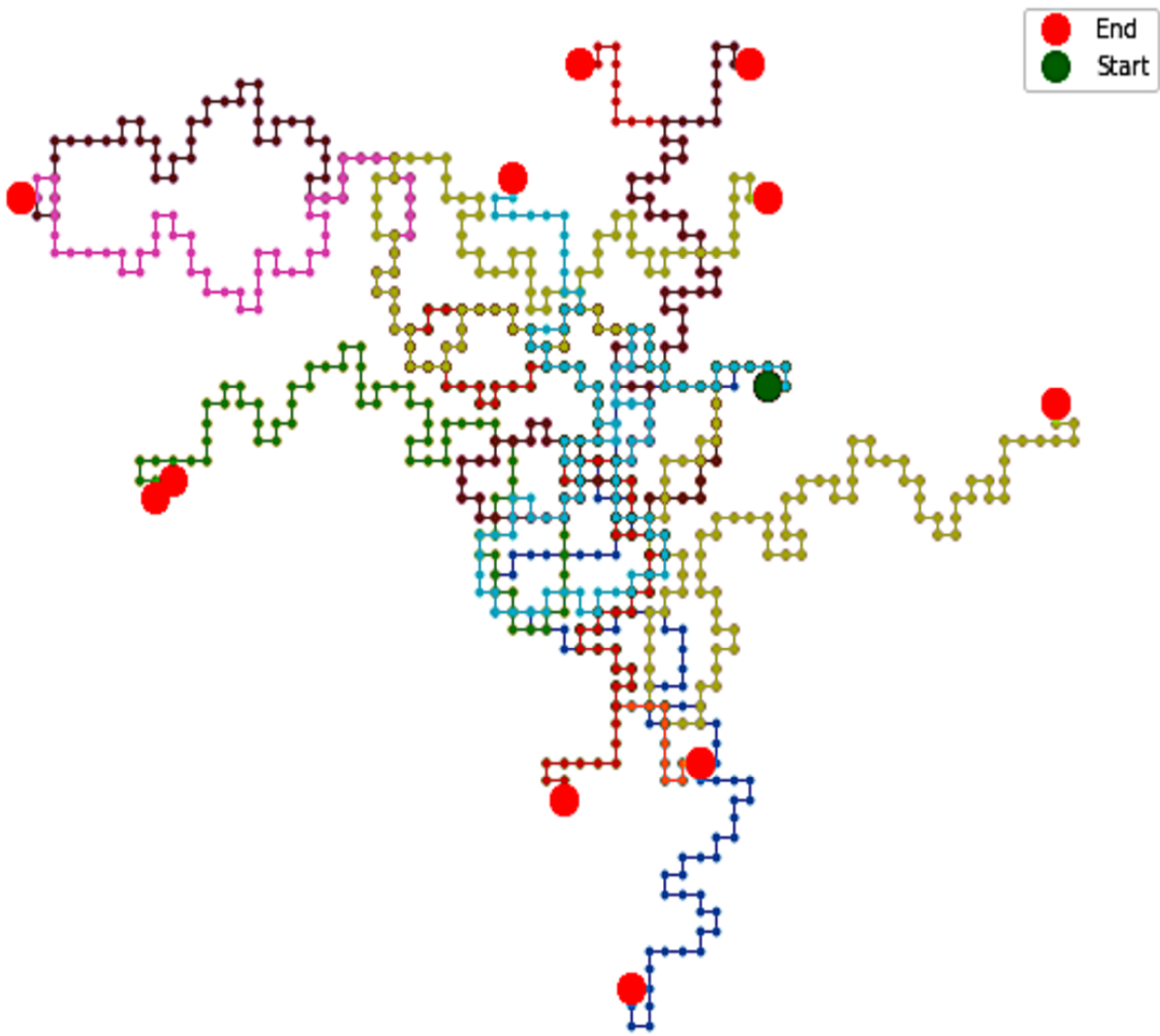

Fig. 6: point-source, 20 rays, length 100, uniform recursive $\mathcal{O}\left(20^{\log\left(20^{0.17}\right) + 0.72}\right)$ self-avoiding walk.

For nd-dimensional random $Y^{(t)} = Y(t)$ driven $m \times nd$ diffusion $b = (b_{j\xi_\nu})$, the lattice consisting finite semi–infinite $[0, +\infty)$ rays emanating from a single vertex is shown in Figures $1, 2, 3, 5$, for genus $g = 0, 1$, on surface diffusion process $c = (c_\eta) = (c(t, X_\eta(t))), \eta = 1, \ldots, m \mid [0, +\infty) \times \Omega \longrightarrow \mathbb{R}^m$:

$$\left\{ \mathrm{d} \log c_\eta(t, X_\eta(t)) = \left(\frac{\mathrm{d}}{\mathrm{dt}} \log \mathbb{E}[X_\eta(t)] \right) \mathrm{dt} + \sum_{\xi=1}^{n} \sum_{\nu=1}^{d} \underbrace{\sqrt{\frac{1}{2} \left(\frac{\mathrm{d}}{\mathrm{d}\tau} \mathrm{Var}[X_{j\xi_\nu}(\tau)] + \frac{\mathrm{d}}{\mathrm{d}\tau} \mathrm{Var}[X_{j\xi_\nu}(\tau)] \,\Big|_{\tau = t_0} \right)}}_{\gamma_{j\xi_\nu}(\tau)} \mathrm{d}Y_{\xi_\nu}(t) \;\; \Big| X_\eta(t) = \mathbb{E}[X_\eta(t)] \right\}$$

corresponding to stopping-time t solution, starting at state $(X_\eta^{(0)} = X_\eta(t_0))$:

$$\left(c_\eta(t, X_\eta(t)) = X_\eta^{(0)} e^{X_\eta(t)} \;\Big|\; X_\eta(t) = \mathbb{E}[X_\eta(t)];\; X_\eta(t_0) = \mathbb{E}[X_\eta(t)] \,\big|_{t=t_0} = \mathbb{E}[X_\eta(t_0)] \geqslant 1,\; \forall\, t_0 \geqslant 0 \right)$$

where $\dfrac{\mathrm{d\,X}_\eta(\mathrm{t})}{X_\eta(t)}$, $\mathrm{d} \log X_\eta(t)$ are instantaneous resp. geometric rate; $\mathrm{d}\,\mathrm{Cov}[\log x_{\eta\xi_\nu}(t), \log x_{k\xi_\nu}(t)] = \sum_{\xi=1}^{n} \sum_{\nu=1}^{d} \gamma_{\eta\xi_\nu}(t)\, \gamma_{k\xi_\nu}(t)\, \mathrm{dt}$

$$\mathrm{d}X_\eta(t) = \left(\frac{\mathrm{d}}{\mathrm{dt}} \log \mathbb{E}[X_\eta(t)] + \frac{1}{2} \sum_{\xi=1}^{n} \sum_{\nu=1}^{d} \gamma^2_{\eta\xi_\nu}(t) \right) \mathbb{E}[X_\eta(t)]\, \mathrm{dt} + \mathbb{E}[x_\eta(t)] \sum_{\xi=1}^{n} \sum_{\nu=1}^{d} \gamma_{\eta\xi_\nu}(t)\, \mathrm{d}Y_{\xi_\nu}(t) \;\Big|\, X_\eta(t_0) = \mathbb{E}[X_\eta(t)] \,\big|_{t=t_0}$$

$$X_\eta(t) = \mathbb{E}[X_\eta(t)] = \mathbb{E}[X_\eta(t)] \,\big|_{t=t_0} + \int_{t_0}^{t} \left(\frac{\mathrm{d}}{\mathrm{d}\tau} \log \mathbb{E}[X_\eta(\tau)] + \frac{1}{2} \sum_{\xi=1}^{n} \sum_{\nu=1}^{d} \gamma^2_{\eta\xi_\nu}(\tau) \right) \mathbb{E}[X_\eta(\tau)]\, \mathrm{d}\tau + \int_{t_0}^{t} \mathbb{E}[x_\eta(\tau)] \sum_{\xi=1}^{n} \sum_{\nu=1}^{d} \gamma_{\eta\xi_\nu}(\tau)\, \mathrm{d}Y_{\xi_\nu}(\tau)$$

$$\mathrm{Var}[x_{\eta\xi_\nu}(t)] = \mathrm{Var}[x_{\eta\xi_\nu}(\min(t,s))] \,\big|_{t=s} = \mathrm{Cov}[x_{\eta\xi_\nu}(t), x_{\eta\xi_\nu}(s)] \,\big|_{t=s} = \mathbb{E}[x_{\eta\xi_\nu}(t)] \cdot \mathbb{E}[x_{\eta\xi_\nu}(s)] \left(-1 + \mathbb{E}^2[x_{\eta\xi_\nu}(\min(t,s))] \right) \big|_{t=s};$$

$$\mathrm{Cov}[x_{\eta\xi_\nu}(t),\, x_{\eta\xi_\nu}(s)] = -\mathbb{E}[x_{\eta\xi_\nu}(t)]\, \mathbb{E}[x_{\eta\xi_\nu}(s)] + \mathbb{E}[x_{\eta\xi_\nu}(t) \cdot \neg\, (x_{\eta\xi_\nu}(t \wedge s))] \cdot \mathbb{E}[x_{\eta\xi_\nu}(s) \cdot \neg\, (x_{\eta\xi_\nu}(s \wedge t))]$$

$$= -\mathbb{E}[x_{\eta\xi_\nu}(t)] \cdot \mathbb{E}[x_{\eta\xi_\nu}(s)] + \mathbb{E}[x_{\eta\xi_\nu}(t) \cdot x_{\eta\xi_\nu}(s)] = -\mathbb{E}[x_{\eta\xi_\nu}(t)] \cdot \mathbb{E}[x_{\eta\xi_\nu}(s)] + \mathbb{E}[x_{\eta\xi_\nu}(t)] \cdot \mathbb{E}[x_{\eta\xi_\nu}(s)] \cdot \mathbb{E}^2[x_{\eta\xi_\nu}(\min(t,s))];$$

for continuous drift $\frac{\mathrm{d}}{\mathrm{dt}} \log \mathbb{E}[X_\eta(t)]$ and continuous root $\sqrt{\mathrm{Var}[X_{\eta\xi_\nu}(t)]}$.

Thus, the wild forest is a continuum diffusion process i.e. $n \times n$ Markov chain, continuous process on countable (finite) index-set scheme of supertree-lattice Gaussian step-drifting, Dirichlet form in large deviation asymptotics. The higher genus $g \geqslant 2$ object (akin to higher-dimension faces glued along lower-dimension boundary in finite chain complex) supports convergence of stratified space for graded Hamiltonian perturbed by noise, with complexity by projective limits of iterated diffusion scheme on special queuing fractals in controlled topology metric spaces; and, all local time t Markov property converges in lattice-asymptotics averaging-principle.

Definition 1.1. *Let Ω be f support; $f(y_{\xi\eta}) dy_{\xi\eta} = \mathbb{P}[Y_{\xi\eta}^{(t)} \in dy_{\xi\eta}] = f(dF(y_{\xi\eta}))$, then*

$$f \geqslant 0, \quad \int_\Omega f(y_{\xi\eta})\, dy_{\xi\eta} = 1, \quad \int_\Omega \lambda_{\xi\eta}\; f(y_{\xi\eta})\, dy_{\xi\eta} \;\in\; \mathbb{F}_\Omega.$$

Lemma 1.1. *The (convex-, quadratic-) covariance Σ for right-continuous Brownian bridge process $\mathcal{N}(\mu, \sqrt{\Sigma})$, stopping times $t \in (0,1)$, states $X_{\xi\wedge t}$, from $X_{\xi\wedge t_{\downarrow\infty}=0} = 0$, to $X_{\xi\wedge t_{\uparrow\infty}=1} = q_\xi \geqslant 0$; $|\xi| < \infty$.*

Proof. Take $X \in \mathbb{C}^n$; $|\xi| = n$; $\xi = 0, \ldots, n-1$; $\mu = (\mu_\xi) \in \mathbb{C}^n$; $\Sigma = \mathrm{Cov}[X] = (\Sigma_{\xi\eta}) \in \mathbb{C}^{n \times n}$ pos. def. sym.

$$\mu_\xi = \mathbb{E}[X_\xi] \equiv \mathbb{E}[X_{\xi\wedge t}] = \mu_{\xi\wedge t} = X_{\xi\wedge t_{\downarrow\infty}} + \frac{(X_{\xi\wedge t_{\uparrow\infty}} - X_{\xi\wedge t_{\downarrow\infty}})(t - t_{\downarrow\infty})}{t_{\uparrow\infty} - t_{\downarrow\infty}}; \quad \frac{1}{\mathrm{Cov}[X_{[n]}]} = \Sigma^{-1}; \quad \Sigma^{-1}_{\xi\eta} = (-1)^{\eta+\xi}\, \frac{\mathrm{Minor}(\Sigma_{\xi\eta})}{\det(\Sigma)};$$

$$\widetilde{\Sigma_{\xi\eta}} = \Sigma_{\xi\eta} \equiv \mathrm{Cov}[X_{\xi\wedge t}, X_{\eta\wedge s}] = \mathbb{E}[(X_{\xi\wedge t} - \mu_{\xi\wedge t}) \otimes (X_{\eta\wedge s} - \mu_{\eta\wedge s})] = \frac{K_{\xi_t\eta_s}\left(\min(s,t) - t_{\downarrow\infty}\right)\left(t_{\uparrow\infty} - \max(s,t)\right)}{t_{\uparrow\infty} - t_{\downarrow\infty}}$$

for all paths ξ, η of the same process; where $s, t \in (0,1)$, and $\mathbb{E}[X^2_{\xi\wedge t}] = t(1-t) K_{\xi_t\xi_t} + t^2 q_\xi^2$. □

Lemma 1.2. *Iff centered, $\mathbb{C}^n$ Brownian bridge process $(X_\xi \equiv X_{\xi\wedge t})$ is invariant of unitary $U \in U(n)$ transformations, where $U(n)$ is the group of $n\times n$ unitary matrices.*

Proof. Set Hermitian $H:\Omega \longrightarrow \mathbb{C}^{n\times n}$ for unitary $U = e^{\sqrt{-1}H}$ by Schur-like decomposition iff Σ is positive-definite, conjugate-symmetric. Then, by $\mathbb{E}[X_{\xi\wedge t}] \equiv U\mathbb{E}[X_{\xi\wedge t}]$; $\sqrt{\Sigma} \equiv \sqrt{\widetilde{U}\,\Sigma\,U}$; $\forall\, y_\xi \in \mathbb{R}$: for all $\mathbb{S}^1$ ball radii $\sqrt{2\Sigma_{\eta,\sigma(\eta)}}$, hyper-surface area $S\big(\mathcal{B}_2(\sqrt{2\Sigma_{\eta,\sigma(\eta)}})\big)$, and i.i.d[2]-like trace:

$$\phi \cong (\phi_{\xi_t}:\Omega \longmapsto \mathbb{R}_+) \;\Big|\, \phi(y) = \int_{\mathbb{R}^n} \exp\Big(\sum_\xi T_{\phi_\xi}\Big) dx_0 \cdots dx_{n-1} = \int_{\mathbb{R}^n} \exp\Big(\text{trace}(T_\phi)\Big) dx;$$

$$T_{\phi_\xi} = \; it\mu_\xi y_\xi - \frac{1}{2}\sum_\eta \Sigma_{\xi\eta}\, y_\xi \otimes y_\eta - \frac{1}{2}\left(\frac{x_\xi - \mu_\xi}{\sqrt{\Lambda_{\xi\xi}}} + r_\xi\big(x_\eta, \forall\, \eta \neq \xi\big)\right)^2 - \frac{1}{2n}\ln\left(\sum_{\sigma\in\mathcal{S}_n} \text{sgn}[\sigma] \prod_{\eta=0}^{n-1} S\big(\mathcal{B}_2(\sqrt{2\Sigma_{\eta,\sigma(\eta)}}\,)\big)\right)$$

where $\max(\Sigma_{\xi\eta})$ is spectral radius for i.i.d X, canonical eigenvectors. And, for $(\mu_\xi) = \vec{0}$, the joint eigenvalue density in terms of Bernoulli and Zeta functions, for eigenvalues $\Lambda_{\xi\xi}$, which equals

$$n!\left(g\big(f_{11<\cdots<nn}\big)\prod_{\xi\leqslant\eta}(\Lambda_{\xi\xi} - \Lambda_{\eta\eta})^2\, dy\right)\Big/\sqrt{\left(\tfrac{2(n!)}{B_n}\right)\,\zeta(2n)\,\det\Sigma} \quad \Big|\; \zeta(2n) = \sum_{q\geqslant 1}\frac{1}{q^{2n}}$$

is an invariant of U, for all $\mu \equiv U\mu$, $\sqrt{\Sigma} \equiv \sqrt{\widetilde{U}\,\Sigma\,U}$, with Hermitian $X_{\xi\wedge t} = a_{\xi\wedge t} + \sqrt{-1}\; b_{\xi\wedge t}$. □

Remark. By $\text{Trace}(T_{\mathbb{P}}) = -\frac{1}{2}\sum_\xi \left(f_{\mathbb{P}_\xi}(\cdot)\big/\sqrt{\rho_{\xi_{t(\xi)}}\cdot\Sigma_{\xi_{t(\xi)}\xi_{t(\xi)}}}\right)^2$, $\forall\, X_{\xi\wedge t} \,|\, \mathscr{X} = (\mathscr{X}_{1_{t(1)}}, \ldots, \mathscr{X}_{n_{t(n)}})$;

$$\Phi(\mathscr{X}) = \int_{-\infty}^{\mathscr{X}_{1_{t(1)}}}\!\!\cdots\int_{-\infty}^{\mathscr{X}_{n_{t(n)}}} \frac{1}{\underbrace{\int_0^\infty\!\!\cdots\int_0^\infty}_{n} \sqrt{2^n\det(\Sigma)}\,\exp\Big(-\sum_{\xi=1}^n \tau_\xi\Big)\prod_{\xi=1}^n \tau_\xi^{\frac{1}{2}-1} d\tau_\xi} \exp\Big(\text{trace}(T_{\mathbb{P}})\Big) dx_0\cdots dx_{n-1}$$

where X is exchangeable i.e. continuous (Σ-nonsingular) by the ϕ argument.

Corollary 1.1 (Maxwell). *$X \in \mathbb{R}^3$ is centered Gaussian if and only if $UX \overset{d}{=} X \mid \widetilde{U}U = U\widetilde{U} = I$.*

Proof. Follows from the prior lemma for all $n \geqslant 1$ by restricting the field $\mathbb{C}$ to $\mathbb{R}$.

Theorem 1.1. *$X = (X_{0\wedge t_0}, \ldots, X_{n-1\wedge t_{n-1}}) \overset{d}{\longrightarrow} (Z_{0\wedge t_0}, \ldots, Z_{n-1\wedge t_{n-1}})$ if X is centered Gaussian, for standard $Z_{\xi\wedge t} \sim \mathcal{N}(0, \sqrt{t(1-t)})$ i.e. centered $(\mu = \widetilde{0})$ and $(\Sigma_{\xi\eta})_{\xi,\eta=0}^{n-1} = t(1-t)I_n$, for all Z.*

Proof. On $\{x \in \mathbb{R}^n : \frac{1}{2}\|x\|^2 = \frac{n}{2}\}$, the vector $(X_{0\wedge t_0}, \ldots, X_{n-1\wedge t_{n-1}})$ is uniformly distributed. Take Z as standard Gaussian on $\mathbb{R}^n$, then induction follows law of large numbers:

$$\mathbb{P}\left(\lim_{n\longrightarrow\infty}\frac{1}{n}X_{0\wedge t_0} = \frac{Z_{0\wedge t_0}}{\sqrt{Z_{0\wedge t_0}^2 + \cdots + Z_{n-1\wedge t_{n-1}}^2}}\right) = 1, \quad \text{i.e. } X_{0\wedge t_0} \xrightarrow{\text{a.s.}} Z_{0\wedge t_0} \quad \text{as } n \longrightarrow \infty. \qquad \square$$

Corollary 1.2 (de Finetti, Shoenberg). *$X = (X_{00}, \ldots, X_{N-1,0}, \ldots, X_{0,k}, \ldots, X_{N-1,k})$ is i.i.d mixture, for $\mathbb{R}$ random $X_{00}, \ldots, X_{N-1,0}, X_{0,1}, \ldots, X_{N-1,1}, \ldots$ iff $UX \overset{d}{=} X$ for all $\widetilde{U}U = U\widetilde{U} = I$.*

Proof. X is clearly exchangeable by bridge lemma. By Maxwell, X is centered Gaussian iff i.i.d i.e. X is mixture of i.i.d random variables. And, by prior theorem, X is distributed as

$$X \overset{\text{d}}{=} SZ \mid S \geqslant 0.$$

[2] where i.i.d $\implies \mathbb{P}(Uu \in \Omega) = \int_\Omega f(y)dy = \int_\Omega \mu(dy) = \int_\Omega \frac{1}{(2\pi)^{nd/2}\sqrt{t^{nd}}\det\Sigma}\exp\Big(-\frac{\langle y - t\mu,\, t^{-1}\Sigma^{-1}(y - t\mu)\rangle}{2}\Big)dy \;\Big|\; \mu \equiv U\mu, \sqrt{\Sigma} \equiv \sqrt{\widetilde{U}\,\Sigma\,U}$

for general Lebesgue measure dx: $\inf\{\sum_{B\in\mathcal{C}}\text{Vol}(B) \,|\, \text{Vol}(B) = \bigtimes_{\gamma=1}^m (b_\gamma - a_\gamma),\ \mathbb{R}^{nd} \ni Y^{(t)} \subseteq \bigcup_{B\in\mathcal{C}} B\}$,

resp. $\inf\{\sum_{\gamma=1}^\infty \nu(I_\gamma) \,|\, \nu(I_\gamma) = b_\gamma - a_\gamma,\ \mathbb{R} \ni Y^{(t)} \subseteq \bigcup_{j=1}^\infty I_\gamma\}$, for countable open $\mathcal{C} = (B = \bigtimes_{\gamma=1}^m (a_\gamma, b_\gamma))_{B\in\mathcal{C}}$, resp. $(I_\gamma = (a_\gamma, b_\gamma))_{\gamma\in\mathbb{N}}$.

Theorem 1.2. *For* $\widetilde{y}\in\mathbb{C}^n\ni X\sim\mathcal{N}(\mu,\sqrt{\Sigma})$; $\mu=(\mu_\xi)\in\mathbb{C}^{n\times 1}$; $\Sigma=(\Sigma_{\xi\eta})\in\mathbb{C}^{n\times n}$; *an entire function:*

$$\text{(i)}\quad \phi(y)=e^{\left\{i\widetilde{y}\mu-\frac{1}{2}\widetilde{y}\Sigma y\right\}}\ \Big|\ \phi(y)=\mathbb{E}[e^{i\widetilde{y}X}]=\mathbb{E}\big[\big(e^{i\widetilde{y}_{t_\xi}X_{t_\xi}}\big)^{\otimes\xi}\big]=\mathbb{E}\big[e^{\left(i\sum_\xi\widetilde{y}_{t_\xi}X_{t_\xi}\right)}\big] \tag{5}$$

$$\text{(ii)}\quad \phi(y)=\mathbb{E}\big[\big(e^{i\widetilde{y}_{t_\xi}X_{t_\xi}}\big)^{\otimes\xi}\big]=\bigotimes_\xi\mathbb{E}\big[e^{i\widetilde{y}_{t_\xi}X_{t_\xi}}\big]=\bigotimes_\xi\sum_{k=0}^\infty\frac{1}{k!}\sum_{k_\Sigma=0\,|\,k=k_\mu+2k_\Sigma}^{\left\lfloor\frac{k}{2}\right\rfloor}\phi_{k,k_\Sigma}(y_\xi)=\bigotimes_\xi\sum_{k=0}^\infty\frac{1}{k!}\phi_k(y_\xi)\ \Big|\ X=i.i.d \tag{6}$$

$$\text{(iii)}\quad \mathbb{E}\big[\big(\widetilde{y}_{t_0}X_{t_0}+\cdots+\widetilde{y}_{t_{2n-1}}X_{t_{2n-1}}\big)^{2n}\big]=\frac{(2n)!}{n!\,2^n}\left(\sum_{\xi=0}^{2n-1}\sum_{\eta=0}^{2n-1}\widetilde{y}_{t_\xi}\widetilde{y}_{t_\eta}\Sigma_{t_\xi t_\eta}\right)^n\ \Big|\ X=centered \tag{7}$$

Proof. (i)

$$\phi(y)=\frac{1}{\sqrt{\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)\prod_{\xi=0}^{n-1}\mathcal{B}_2\big(\sqrt{2\Sigma_{\xi,\sigma(\xi)}}\big)}}\int_{\mathbb{R}^n}e^{\left\{-\frac{1}{2}\widetilde{(x-\mu-i\Sigma y)}\,\Sigma^{-1}(x-\mu-i\Sigma y)-2i\widetilde{y}\mu+\widetilde{y}\Sigma y\right\}}dx\ \Big|\ \begin{array}{l}\operatorname{sgn}(\sigma)=(-1)^{t(\sigma)};\\ t(\sigma):=(\sigma(0),\dots,\sigma(n-1))\\ \qquad\longrightarrow(0,\dots,n-1)\end{array}$$

$$=e^{\left\{i\widetilde{y}\mu-\frac{1}{2}\widetilde{y}\Sigma y\right\}}\ \Big|\ \begin{array}{l}\widetilde{(x-\mu-i\Sigma y)}\,\Sigma^{-1}(x-\mu-i\Sigma y)=\widetilde{(x-\mu)}\,\Sigma^{-1}(x-\mu)-i\,\widetilde{(x-\mu)}(\Sigma^{-1}\Sigma)y-i\,\widetilde{y}(\Sigma\Sigma^{-1})(x-\mu)-\widetilde{y}(\Sigma\Sigma^{-1})\Sigma y\\ \qquad=\widetilde{(x-\mu)}\,\Sigma^{-1}(x-\mu)-2i\,\widetilde{y}x-2i\,\widetilde{y}\mu+\widetilde{y}\Sigma y\end{array}$$

by $\widetilde{x}y=\widetilde{y}x$, for transposes $\widetilde{x},\widetilde{y}$ of complex-conjugates with respect to y and x, respectively, and

$$y^T\Sigma\Sigma^{-1}\Sigma y=\Big(\sum_\xi\sum_\ell\sum_k y_\xi^T\Sigma_{\xi k}\Sigma_{k\ell}^{-1}\Sigma_{\ell,\cdot}\Big)y=\Big(\sum_\xi\sum_\ell y_\xi^T(\delta_{\xi\ell})\Sigma_{\ell,\cdot}\Big)y=\Big(\sum_\xi y_\xi^T\Sigma_{\xi,\cdot}\Big)y=\sum_\eta\sum_\xi y_\xi^T\Sigma_{\xi\eta}y_\eta=y^T\Sigma y$$

moreover,

$$I_n=\Sigma^T\Sigma^{-1}=\Sigma\Sigma^{-1}=\Sigma^{-1}\Sigma=((\Sigma^{-1}\Sigma)_{\xi\eta});\quad(\Sigma^{-1}\Sigma)_{\xi\eta}=\sum_\ell\Sigma_{\xi\ell}^{-1}\Sigma_{\ell\eta}=\delta_{\xi\eta}=\begin{cases}1 & \text{if }\ \xi=\eta\\ 0 & \text{otherwise}\end{cases},\quad\phi(y)=\mathbb{E}[e^{iy^TX}]$$

(ii) Set $0^0\equiv 1$; $\mathbb{E}[X_\xi^0]\equiv 1$; $X^k=(X_\xi^k)$; for Boolean $\mathbb{1}_{k\in(\cdot)}:\mathbb{N}_0\xrightarrow{k/\mathbb{Z}_2}\{0,1\}$, Dirac delta $\delta(x_\xi)$, where $\phi(-iy)=\theta(y)$, $iy\otimes iy=-y\otimes y$, $\widetilde{(y-y(0))}^k=(\widetilde{y_\xi-y_\xi(0)})^k=\widetilde{y}_\xi^k$, about $\widetilde{y}(0)=\vec{0}$. Then, the following bilateral Laplace-Stieltjes series, $\forall\,(\xi,k_\xi\in\mathbb{N}_0;\ \xi,k_\xi<\infty)$:

$$\theta(y)=\mathbb{E}\Big[\sum_{k=0}^\infty\frac{1}{k!}\Big(\widetilde{y}\Big(\frac{\partial}{\partial\widetilde{y}}e^{\widetilde{y}X}\Big|_{\widetilde{y}=\widetilde{y}(0)=\vec{0}}\Big)\Big)^k\Big]=\mathbb{E}\Big[\sum_{k=0}^\infty\frac{1}{k!}\Big(\sum_\xi\Big(\widetilde{y}_{t_\xi}\Big(\frac{\partial}{\partial\widetilde{y}_{t_\xi}}e^{\left(\sum_\xi\widetilde{y}_{t_\xi}X_{t_\xi}\right)}\Big|_{\widetilde{y}=\widetilde{y}(0)=\vec{0}}\Big)\Big)\Big)^k\Big]=\sum_{k=0}^\infty\frac{1}{k!}\mathbb{E}\Big[\Big(\sum_\xi\widetilde{y}_{t_\xi}X_{t_\xi}\Big)^k\Big]$$

$$=\sum_{k=0}^\infty\frac{1}{k!}\sum_{k=k_0+\cdots+k_{n-1}}\binom{k}{k_0,\dots,k_{n-1}}\mathbb{E}\big[X_{t_0}^{k_0}\cdots X_{t_{n-1}}^{k_{n-1}}\big]\,\underbrace{\widetilde{y}_{t_0}\otimes\cdots\otimes\widetilde{y}_{t_0}}_{k_0\ \text{times}}\otimes\cdots\otimes\underbrace{\widetilde{y}_{t_{n-1}}\otimes\cdots\otimes\widetilde{y}_{t_{n-1}}}_{k_{n-1}\ \text{times}}\ \Big|\ \widetilde{y}_{t_\xi}^{k_\xi}=\underbrace{\widetilde{y}_{t_\xi}\otimes\cdots\otimes\widetilde{y}_{t_\xi}}_{k_\xi\ \text{times}}$$

where

$$\mathbb{E}\big[\big(\widetilde{y}_{t_\xi}^{k_\xi}X_{t_\xi}^{k_\xi}\big)^{\otimes\xi}\big]=\Bigg(\sum_{k_\Sigma=0\,|\,k_\xi=k_\mu+2k_\Sigma}^{\left\lfloor\frac{k_\xi}{2}\right\rfloor}\widetilde{\theta}_{k_\xi,k_\Sigma}\,\big(\widetilde{y}_{t_\xi}\mu_{t_\xi}\big)^{k_\mu}\big(\Sigma_{t_\xi t_\xi}\widetilde{y}_{t_\xi}\otimes y_{t_\xi}\big)^{k_\Sigma}\Bigg)^{\otimes\xi}\ \Bigg|\ \widetilde{\theta}_{k_\xi,\frac{k_\Sigma}{2}}=\Gamma\Big(\frac{k_\Sigma+1}{2}\Big)\frac{\big(1+(-1)^{k_\Sigma}\big)}{2^{\left(\frac{2-k_\Sigma}{2}\right)}}\frac{1}{\sqrt{\pi}}\binom{k_\xi}{k_\Sigma}$$

by

$$\mathbb{E}\big[\big(e^{\widetilde{y}_{t_\xi}X_{t_\xi}}\big)^{\otimes\xi}\big]=\Bigg(\sum_{k_\xi=0}^\infty\frac{1}{k_\xi!}\widetilde{y}_{t_\xi}^{k_\xi}\,\mathbb{E}\big[X_{t_\xi}^{k_\xi}\big]\Bigg)^{\otimes\xi}=\Bigg(\sum_{k_\xi=0}^\infty\frac{1}{k_\xi!}\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\Big(\widetilde{y}_{t_\xi}\mu_{t_\xi}+\big(2\Sigma_{t_\xi t_\xi}\widetilde{y}_{t_\xi}\otimes y_{t_\xi}\big)^{\frac{1}{2}}x_\xi\Big)^{k_\xi}e^{-x_\xi^2}\,dx_\xi\Bigg)^{\otimes\xi}$$

$$=\Bigg(\sum_{k_\xi=0}^\infty\frac{1}{k_\xi!}\sum_{k_\Sigma=0\,|\,k_\xi=(k_\mu+2k_\Sigma)}^{\left\lfloor\frac{k_\xi}{2}\right\rfloor}2^{k_\Sigma}\,\Gamma\Big(\frac{2k_\Sigma+1}{2}\Big)\frac{\big(1+(-1)^{2k_\Sigma}\big)}{2}\frac{1}{\sqrt{\pi}}\binom{k_\xi}{2k_\Sigma}\big(\widetilde{y}_{t_\xi}\mu_{t_\xi}\big)^{k_\mu}\big(\Sigma_{t_\xi t_\xi}\widetilde{y}_{t_\xi}\otimes y_{t_\xi}\big)^{k_\Sigma}=\sum_{k_\xi=0}^\infty\frac{1}{k_\xi!}\sum_{k_\Sigma=0\,|\,k_\xi=k_\mu+2k_\Sigma}^{\left\lfloor\frac{k_\xi}{2}\right\rfloor}\theta_{k_\xi,k_\Sigma}(y_{t_\xi})\Bigg)^{\otimes\xi}$$

$$=\Bigg(\sum_{k_\xi=0}^\infty\frac{1}{k_\xi!}\Bigg(\sum_{k_\Sigma=0\,|\,k_\xi=2(k_\mu+k_\Sigma)}^{\left(\frac{k_\xi}{2}\right)}\widetilde{\theta}_{k,k_\Sigma}\big(\widetilde{y}_{t_\xi}\mu_{t_\xi}\big)^{2k_\mu}\big(\Sigma_{t_\xi t_\xi}\widetilde{y}_{t_\xi}\otimes y_{t_\xi}\big)^{k_\Sigma}\,\mathbb{1}_{k_\xi\in 2\mathbb{Z}}+\sum_{k_\Sigma=0\,|\,k_\xi=2(k_\mu+k_\Sigma)+1}^{\left\lfloor\frac{k_\xi}{2}\right\rfloor}\widetilde{\theta}_{k,k_\Sigma}\big(\widetilde{y}_{t_\xi}\mu_{t_\xi}\big)^{2k_\mu+1}\big(\Sigma_{t_\xi t_\xi}\widetilde{y}_{t_\xi}\otimes y_{t_\xi}\big)^{k_\Sigma}\,\mathbb{1}_{k_\xi\in 2\mathbb{Z}+1}\Bigg)\Bigg)^{\otimes\xi}$$

$$\frac{\left(1+(-1)^{k_\Sigma}\right)\sqrt{u}}{2^{\left(\frac{1-k_\Sigma}{2}\right)}}\,\Gamma\Big(\frac{k_\Sigma+1}{2}\Big) = \int_{\mathbb{R}} x_\xi^{k_\Sigma} e^{-x_\xi^2/u}\,dx_\xi = \frac{k_\Sigma!\,\left(1+(-1)^{k_\Sigma}\right)\sqrt{u}}{\left(\frac{k_\Sigma}{2}\right)!\;2^{\left(\frac{k_\Sigma+1}{2}\right)}}\int_0^\infty e^{-x_\xi}\,x_\xi^{\frac{1}{2}-1}dx_\xi = \frac{\left(1+(-1)^{k_\Sigma}\right)\sqrt{u\pi}}{\left(\frac{k_\Sigma}{2}\right)!\;2^{\left(\frac{k_\Sigma+1}{2}\right)}}\int_{\mathbb{R}}\frac{d^{k_\Sigma}}{dx_\xi^{k_\Sigma}}\,\delta(x_\xi)\,dx_\xi$$

that is, for all i.i.d $X=(X_\xi)_{\xi=0}^{n-1}$, i.e. $\Sigma_{\xi\eta}=\mathrm{Cov}(X_\xi,X_\eta)=0,\ \forall\,\xi\neq\eta=1,\ldots,n$, then $\phi(y)=\theta(iy)$:

$$e^{\left\{\sum_{\xi=0}^{n-1}\left(i\,\widetilde{y}_\xi\mu_\xi-\frac{1}{2}\Sigma_{\xi\xi}\,\widetilde{y}_\xi\otimes y_\xi\right)\right\}} = \prod_{\xi=0}^{n-1}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{k_\Sigma=0\,|\,k=(k_\mu+2k_\Sigma)}^{\left\lfloor\frac{k}{2}\right\rfloor}(2k_\Sigma-1)!!\binom{k}{2k_\Sigma}\left(i\,\widetilde{y}_\xi\mu_\xi\right)^{k_\mu}(-1)^{k_\Sigma}\left(\Sigma_{\xi\xi}\,y_\xi\otimes y_\xi\right)^{k_\Sigma}.$$

And,

$$\mathbb{E}\Big[\prod_{\xi=0}^{n-1}\widetilde{y}_{t_\xi}^{k_\xi}X_{t_\xi}^{k_\xi}\Big] = \mathbb{E}\big[\widetilde{y}_{t_0}^{k_0}X_{t_0}^{k_0}\cdots\widetilde{y}_{t_{n-1}}^{k_{n-1}}X_{t_{n-1}}^{k_{n-1}}\big] = \mathbb{E}\big[\underbrace{\widetilde{y}_{t_0}X_{t_0}\cdots\widetilde{y}_{t_0}X_{t_0}}_{k_0\ \text{times}}\cdots\underbrace{\widetilde{y}_{t_{n-1}}X_{t_{n-1}}\cdots\widetilde{y}_{t_{n-1}}X_{t_{n-1}}}_{k_{n-1}\ \text{times}}\big]$$

$$=\prod_{\xi=0}^{n-1}\widetilde{y}_{t_\xi}^{k_\xi}\,\mathbb{E}[X_{t_\xi}^{k_\xi}] + \left(\sum_{n=1}^{\left\lfloor\frac{1}{2}\sum_{\xi=0}^{n-1}k_\xi\right\rfloor}\left(\sum_{\{(\xi_0<\eta_0),\ldots,(\xi_{n-1}<\eta_{n-1})\}}\Sigma_{t_{\xi_0}t_{\eta_0}}\cdots\Sigma_{t_{\xi_{n-1}}t_{\eta_{n-1}}}\prod_{\ell=0}^{n-1}\mathbb{E}\big[X_{t_\ell}^{k_\ell-|\ell\in\{(\xi_0<\eta_0),\ldots,(\xi_{n-1}<\eta_{n-1})\}|}\big]\right)\right)\prod_{\xi=0}^{n-1}\widetilde{y}_{t_\xi}^{k_\xi}$$

for sum over all disjoint $\{(\xi_0<\eta_0),\ldots,(\xi_{n-1}<\eta_{n-1})\}$, number $|\ell\in\{(\xi_0<\eta_0),\ldots,(\xi_{n-1}<\eta_{n-1})\}|$ of ℓ in partition $\{(\xi_0<\eta_0),\ldots,(\xi_{n-1}<\eta_{n-1})\}$. (iii) follows from the above equality. □

Remark. In convergence $\mathbf{P}\Big(\frac{X_{1k}}{\sqrt{k}}\leqslant x_{1k},\ldots,\frac{X_{Nk}}{\sqrt{k}}\leqslant x_{Nk}\Big)\xrightarrow{k\longrightarrow\infty}(\Phi(x_{1,k}))^{\otimes N}\ \Big|\ X_{\xi k}=\sum_{\eta=1}^{k}X_{\xi\eta},\ \forall\,k\in\mathbb{N}$, and i.i.d $X_{\xi\eta}\sim\mathcal{N}\big(0,\sqrt{\Sigma_{\xi\xi}}\,\big)$, i.e. $\mathbb{E}[X_{\xi\eta}]=\mu_\xi=0$, $\mathrm{Var}[X_{\xi\eta}]=\Sigma_{\xi\xi}$, $\forall\,\xi,\eta$; $0^0=1$; then

$$\phi(y) = \prod_{\xi}\sum_{k=0}^{\infty}\frac{1}{k!}\,\widetilde{y_\xi^k}\,\mathbb{E}[X_\xi^k] = \prod_{\xi=1}^{N}\sum_{k=0}^{\infty}\frac{1}{k!}\,\phi_{k,\frac{k}{2}}(y_\xi)\,\mathbb{1}_{k\,\in\,2\mathbb{Z}_{\geqslant 0}} = \prod_{\xi=1}^{N}\sum_{k=0}^{\infty}\frac{1}{k!}\,2^{\left(\frac{k}{2}\right)}\frac{1}{\sqrt{\pi}}\,\Gamma\Big(\frac{k+1}{2}\Big)\,(-1)^{\frac{k}{2}}\left(\Sigma_{\xi\xi}\,y_\xi\otimes y_\xi\right)^{\frac{k}{2}}\,\mathbb{1}_{k\,\in\,2\mathbb{Z}_{\geqslant 0}}$$

$$=\prod_{\xi=1}^{N}\sum_{k=0}^{\infty}\frac{1}{\left(\frac{k}{2}\right)!\;2^{\left(\frac{k}{2}\right)}}\,(-1)^{\frac{k}{2}}\left(t\Sigma_{\xi\xi}\,y_\xi\otimes y_\xi\right)^{\frac{k}{2}}\,\mathbb{1}_{k\,\in\,2\mathbb{Z}_{\geqslant 0}} = \prod_{\xi=1}^{N}\sum_{k=0}^{\infty}(-1)^k\,\frac{(\Sigma_{\xi\xi})^k\,\left(y_\xi\otimes y_\xi\right)^k}{k!\;2^k}$$

$$=\Big(\lim_{k\longrightarrow\infty}\Big\{\Big(1-\frac{1}{2}\frac{\Sigma_{\xi\xi}}{k}+\mathcal{O}\Big(\frac{1}{k}\Big)\Big)\left(y_\xi\otimes y_\xi\right)\Big\}^k\Big)^N = \Big(\lim_{k\longrightarrow\infty}\vartheta_k\,y^{\otimes 2k}\Big)^N = \Big(\lim_{k\longrightarrow\infty}\big(\varrho_k(y)\big)^{\otimes 2k}\Big)^N.$$

In addition, this conclusion follows directly from $\prod_{\xi<\infty}\exp\big\{\langle y_\xi,\,t\mu_\xi\rangle+\frac{1}{2}\langle y_{\xi\eta},\,t\Sigma_{\xi\xi}y_\xi\rangle^{\otimes}\big\}$.

Remark. All 1-dimensional, i.e. $|\xi|=|\eta|=1$, $\mathcal{N}(1,1)$, kth moment sequence rows as follows.

Tab. 1: "Even" and "odd" structure $((\theta_{k,k_\Sigma})\mid y=\vec{1};\,1\otimes 1=1),\ \forall\,k_\Sigma=0,1,\ldots,\left\lfloor\frac{k}{2}\right\rfloor\ \mid\ k=0,1,\ldots,16$

{1}
{1}
{1, 1}
{1, 3}
{1, 6, 3}
{1, 10, 15}
{1, 15, 45, 15}
{1, 21, 105, 105}
{1, 28, 210, 420, 105}
{1, 36, 378, 1260, 945}
{1, 45, 630, 3150, 4725, 945}
{1, 55, 990, 6930, 17325, 10395}
{1, 66, 1485, 13860, 51975, 62370, 10395}
{1, 78, 2145, 25740, 135135, 270270, 135135}
{1, 91, 3003, 45045, 315315, 945945, 945945, 135135}
{1, 105, 4095, 75075, 675675, 2837835, 4729725, 2027025}
{1, 120, 5460, 120120, 1351350, 7567560, 18918900, 16216200, 2027025}

Tab. 2: "Even" structure $((\theta_{k,k_\Sigma}\cdot 2^{k-k_\Sigma})\mid y=\vec{1};\,1\otimes 1=1),\ \forall\,k_\Sigma=0,1,\ldots,\left\lfloor\frac{k}{2}\right\rfloor\ \mid\ k/2=0,1,\ldots,16$

{1}
{2, 1}
{4, 12, 3}
{8, 60, 90, 15}
{16, 224, 840, 840, 105}
{32, 720, 5040, 12600, 9450, 945}
{64, 2112, 23760, 110880, 207900, 124740, 10395}
{128, 5824, 96096, 720720, 2522520, 3783780, 1891890, 135135}
{256, 15360, 349440, 3843840, 21621600, 60540480, 75675600, 32432400, 2027025}
{512, 39168, 1175040, 17821440, 147026880, 661620960, 1543782240, 1654052400, 620269650, 34459425}
{1024, 97280, 3720960, 74419200, 846518400, 5587021440, 20951330400, 41902660800, 39283744500, 13094581500, 654729075}
{2048, 236544, 11235840, 286513920, 4297708800, 39109150080, 215100325440, 691393903200, 1209939330600, 1008282775500, 302484832650, 13749310575}
{4096, 565248, 32643072, 1033697280, 19769460480, 237233525760, 1799020903680, 8481098545920, 23853089660400, 37104806138400, 27828604603800, 7589619437400, 316234143225}
{8192, 1331200, 91852800, 3536332800, 83987904000, 1285014931200, 12850149312000, 83525970528000, 344544628428000, 861361571070000, 1205906199498000, 822208772385000, 205552193096250, 7905853580625}
{16384, 3096576, 251596800, 11573452800, 334183449600, 6349485542400, 80955940665600, 693908062848000, 3946602107448000, 14470874393976000, 32559467386446000, 41439322128204000, 25899576330127500, 5976825306952500, 213458046676875}
{32768, 7127040, 673505280, 36481536000, 1258612992000, 29073960115200, 460337701824000, 5030833455648000, 37731250917360000, 190752435193320000, 629483036137956000, 1287578937554910000, 1502175427147395000, 866639669508112500, 185708500608881250, 6190283353629375}
{65536, 16252928, 1767505920, 111352872960, 4523710464000, 124854408806400, 2403447369523200, 32618214300672000, 311911674250176000, 2079411161667840000, 9461320785588672000, 28383962356766016000, 53219929418936280000, 57313770143469840000, 30703805434001700000, 6140761086800340000, 191898783962510625}

Tab. 3: “Odd” structure $((\theta_{k,k_\Sigma}\cdot 2^{k-k_\Sigma}) \mid y=\vec{1};\ 1\otimes 1=1),\ \forall\, k_\Sigma=0,1,\dots,\left\lfloor \frac{k}{2}\right\rfloor \ \mid\ (k-1)/2=0,1,\dots,16$

{1}
{2, 3}
{4, 20, 15}
{8, 84, 210, 105}
{16, 288, 1512, 2520, 945}
{32, 880, 7920, 27720, 34650, 10395}
{64, 2496, 34320, 205920, 540540, 540540, 135135}
{128, 6720, 131040, 1201200, 5405400, 11351340, 9459450, 2027025}
{256, 17408, 456960, 5940480, 40840800, 147026880, 257297040, 183783600, 34459425}
{512, 43776, 1488384, 26046720, 253955520, 1396755360, 4190266080, 6285399120, 3928374450, 654729075}
{1024, 107520, 4596480, 104186880, 1367452800, 10666131840, 48886437600, 125707982400, 164991726900, 91662070500, 13749310575}
{2048, 259072, 13601280, 387636480, 6589820160, 69193111680, 449755225920, 1766895530400, 3975514943400, 4638100767300, 2319050383650, 316234143225}
{4096, 614400, 38860800, 1360128000, 29072736000, 395389209600, 3459655584000, 19275223968000, 66258582390000, 132517164780000, 139143023019000, 63246828645000, 7905853580625}
{8192, 1437696, 107827200, 4546713600, 119351232000, 2040906067200, 23130268761600, 173477015712000, 845700451596000, 2584084713210000, 4651352483778000, 4439927370879000, 1849969737866250, 213458046676875}
{16384, 3325952, 291852288, 14592614400, 461491430400, 9691320038400, 138101310547200, 1341555588172800, 8803958547384000, 38150487038664000, 104913839356326000, 171677191673988000, 150217542714739500, 57775977967207500, 6190283353629375}
{32768, 7618560, 773283840, 45237104640, 1696391424000, 42918703027200, 751077302976000, 9173872772064000, 77977918562544000, 454871191614840000, 1773997647297876000, 4434994118244690000, 6652491177367035000, 5373165950950297500, 1918987839625106250, 191898783962510625}
{65536, 17301504, 2011299840, 136097955840, 5971297812480, 179138934374400, 3776845866393600, 56652687995904000, 605475602956224000, 4574704555669248000, 24017198917263552000, 85151887070298048000, 195139741202766360000, 270193487819214960000, 202645115864411220000, 67548371954803740000, 6332659870762850625}

Remark. Table 1 and 3 are Bessel resp. Gauss-Hermite polynomials. Table 1, row k generates k_Σ-matchings $\theta_{k,k_\Sigma}(1)$ of complete graph K_k; e.g., $\theta_{2k,k}(1)$ are perfect-matchings (Godsil, 1981).

Derivation. For Gaussian, $\mathbb{R}^{nd}$-valued $(Y^{(t)}_{nd})$, $\forall\,(x_{1_1},\dots,x_{n_d})=x\in\mathbb{R}^{nd}$,

$$\mathbb{E}[Y^{(t)}_{1_1}x_{1_1}\otimes\cdots\otimes Y^{(t)}_{2n_d}x_{2n_d}] = \bigotimes_{\xi=1}^{n}\bigotimes_{\nu=1}^{d}\sum_{\sigma\,\in\,\mathcal{S}_{2nd}/\mathcal{S}_{\frac{nd}{2}}\times\mathcal{S}_2^{nd}}\Big(\mu^{(t)}_{\sigma_{\xi_{2\nu-1}}}\mu^{(t)}_{\sigma_{\xi_{2\nu}}}+\Sigma^{(t)}_{\sigma_{\xi_{2\nu-1}}\sigma_{\xi_{2\nu}}}\Big)\,x_{\sigma_{\xi_{2\nu-1}}}\otimes x_{\sigma_{\xi_{2\nu}}}\ \Bigg|\ \begin{array}{l}\sigma\colon (1,\dots,2nd)\longrightarrow\\ \quad((\sigma_{1_1}<\sigma_{1_2}),\dots,(\sigma_{n_{2d-1}}<\sigma_{n_{2d}}))\end{array}$$

where $(2nd)!\,\mathbb{E}[Y^{(t)}_{1_1}\cdots Y^{(t)}_{2n_d}]$ is the coefficient of $x_{1_1}\otimes\cdots\otimes x_{2n_d}$ in

$$\mathbb{E}\Bigg[\bigg(\sum_{\xi=1}^{2n}\sum_{\nu=1}^{d}Y^{(t)}_{\xi_\nu}x_{\xi_\nu}\bigg)^{2nd}\Bigg]=\frac{(2nd)!}{(nd)!\,2^{nd}}\bigg(\sum_{\xi=1}^{2n}\sum_{\eta=1}^{2n}\sum_{\nu=1}^{d}\sum_{\omega=1}^{d}\mathbb{E}[Y^{(t)}_{\xi_\nu}Y^{(t)}_{\eta_\omega}]x_{\xi_\nu}\otimes x_{\eta_\omega}\bigg)^{nd}=\frac{(2nd)!}{(nd)!\,2^{nd}}\bigg(\sum_{\xi=1}^{2n}\sum_{\eta=1}^{2n}\sum_{\nu=1}^{d}\sum_{\omega=1}^{d}\Big(\mu^{(t)}_{\xi_\nu}\mu^{(t)}_{\eta_\omega}+\Sigma^{(t)}_{\xi_\nu\eta_\omega}\Big)x_{\xi_\nu}\otimes x_{\eta_\omega}\bigg)^{n}.$$

Theorem. For Pochhammer $(a)_m$, in hypergeometric form, $\theta(z)$ equals

$$\bigotimes_{\xi=1}^{n}\bigotimes_{\nu=1}^{d}\sum_{\gamma=0}^{\infty}\frac{1}{\gamma!}\left(\begin{array}{c}2^{\frac{\gamma}{2}}\,\dfrac{1}{\sqrt{\pi}}\,\Gamma\Big(\dfrac{\gamma+1}{2}\Big)\,t^{\frac{\gamma}{2}}\,(\Sigma_{\xi_\nu\xi_\nu})^{\frac{\gamma}{2}}\ \widetilde{\mathcal{H}}\Big(-\dfrac{t\mu^2_{\xi_\nu}}{2\Sigma_{\xi_\nu\xi_\nu}};-\dfrac{\gamma}{2},\dfrac{1}{2}\Big)\,x^{\gamma}_{\xi_\nu}\,\mathbb{1}^{2\mathbb{Z}}_{\gamma}\\ +\\ 2^{\frac{\gamma+1}{2}}\,\dfrac{1}{\sqrt{\pi}}\,\Gamma\Big(\dfrac{\gamma}{2}+1\Big)\,t^{\frac{\gamma-1}{2}+1}\,\mu_{\xi_\nu}\,(\Sigma_{\xi_\nu\xi_\nu})^{\frac{\gamma-1}{2}}\ \widetilde{\mathcal{H}}\Big(-\dfrac{t\mu^2_{\xi_\nu}}{2\Sigma_{\xi_\nu\xi_\nu}};\dfrac{-\gamma+1}{2},\dfrac{3}{2}\Big)\,x^{\gamma}_{\xi_\nu}\,\mathbb{1}^{2\mathbb{Z}+1}_{\gamma}\end{array}\right)$$

where

$$\widetilde{\mathcal{H}}(x;a,b)=\sum_{m=0}^{\infty}\frac{(a)_m}{(b)_m}\frac{x^m}{m!}\ ,\qquad (a)_m=\frac{\Gamma(a+m)}{\Gamma(a)}=\begin{cases}0^0:\equiv 1 & \text{if } m=0\\ \displaystyle\prod_{\ell=0}^{m-1}(a+\ell) & \text{if } m>0.\end{cases}$$

Proof. Follows by expanding the Gauss-hypergeometric series.

Corollary. $\theta(z)$ equals

$$\bigotimes_{\xi=1}^{n}\bigotimes_{\nu=1}^{d}\sum_{\gamma=0}^{\infty}\frac{1}{\gamma!}\left(\begin{array}{c}\left(\begin{array}{c}-\big(-1+e^{i\gamma\pi}\big)\,2^{\frac{\gamma-1}{2}}\,\dfrac{1}{\sqrt{\pi}}\,\Gamma\Big(\dfrac{\gamma}{2}+1\Big)\,t^{\frac{\gamma+1}{2}}\,\mu_{\xi_\nu}x_{\xi_\nu}\,(\Sigma_{\xi_\nu\xi_\nu}x_{\xi_\nu}\otimes x_{\xi_\nu})^{\frac{\gamma-1}{2}}\ \widetilde{\mathcal{H}}\Big(-\dfrac{t\mu^2_{\xi_\nu}}{2\Sigma_{\xi_\nu\xi_\nu}};\dfrac{-\gamma+1}{2},\dfrac{3}{2}\Big)\\ +\\ \big(1+e^{i\gamma\pi}\big)\,2^{\frac{\gamma}{2}-1}\,\dfrac{1}{\sqrt{\pi}}\,\Gamma\Big(\dfrac{\gamma+1}{2}\Big)\,t^{\frac{\gamma}{2}}\,(\Sigma_{\xi_\nu\xi_\nu}x_{\xi_\nu}\otimes x_{\xi_\nu})^{\frac{\gamma}{2}}\ \widetilde{\mathcal{H}}\Big(-\dfrac{t\mu^2_{\xi}}{2\Sigma_{\xi\xi}};-\dfrac{\gamma}{2},\dfrac{1}{2}\Big)\end{array}\right)\mathbb{1}^{2\mathbb{Z}}_{\gamma}\\ +\\ \left(\begin{array}{c}\Big(-1+(-1)^{\gamma-1}\Big)\,2^{\frac{\gamma-2}{2}}\,\dfrac{1}{\sqrt{\pi}}\,\Gamma\Big(\dfrac{\gamma+1}{2}\Big)\,t^{\frac{\gamma}{2}}\,(\Sigma_{\xi_\nu\xi_\nu}x_{\xi_\nu}\otimes x_{\xi_\nu})^{\frac{\gamma}{2}}\ \widetilde{\mathcal{H}}\Big(-\dfrac{t\mu^2_{\xi_\nu}}{2\Sigma_{\xi_\nu\xi_\nu}};-\dfrac{\gamma}{2},\dfrac{1}{2}\Big)\\ +\\ \Big(-1+(-1)^{\gamma-1}\Big)\,2^{\frac{\gamma-1}{2}}\,\dfrac{1}{\sqrt{\pi}}\,\Gamma\Big(\dfrac{\gamma+2}{2}\Big)\,t^{\frac{\gamma+1}{2}}\,\mu_{\xi_\nu}x_{\xi_\nu}\,(\Sigma_{\xi_\nu\xi_\nu}x_{\xi_\nu}\otimes x_{\xi_\nu})^{\frac{\gamma-1}{2}}\ \widetilde{\mathcal{H}}\Big(-\dfrac{t\mu^2_{\xi_\nu}}{2\Sigma_{\xi_\nu\xi_\nu}};\dfrac{-\gamma+1}{2},\dfrac{3}{2}\Big)\end{array}\right)\mathbb{1}^{2\mathbb{Z}+1}_{\gamma}\end{array}\right)$$

where $-1+e^{-i2\gamma\pi}=1+e^{-i(2\gamma+1)\pi}=1+e^{i(2\gamma+1)\pi}=-1+e^{i2\gamma\pi}=0,\ \forall\,\gamma\in\mathbb{Z}\backslash\{0\}$.

Proof. Follows from the theorem. □

Remark. The definition of $x_{\xi_\nu}\otimes x_{\xi_\nu}$ symmetry determines other corollaries.

Corollary. Let an $\mathbb{R}^{nd}$ Gaussian, $\phi(x)$ be entire if

$$\phi(x=(x_{1_1},\ldots,x_{n_d})) = \int_{\mathbb{R}^{nd}} e^{(\langle ix,y\rangle)}\mu(dy) = \bigotimes_{\xi=1}^{n}\bigotimes_{\nu=1}^{d} e^{P_{m_{\xi_\nu}}(x_{\xi_\nu})} \;\Big|\, x_{\xi_\nu}\in\mathbb{C}$$

for all degree m_{ξ_ν} polynomial $P_{m_{\xi_\nu}}(x_{\xi_\nu})$, then $\max(\deg(P_{m_{\xi_\nu}}(x_{\xi_\nu})))\ngtr 2$.

Proof. Following the above, $\phi(-ix)=\exp\Big(\langle x,t\mu\rangle+\frac{1}{2}\langle x,t\Sigma x\rangle\Big)$, $x_{\xi_\nu}\in\mathbb{C}$. Moreover, $\forall\, m\,|\,\beta=r\cos\theta$, $\gamma=r\sin\theta$, then $a_m\cos(m\theta)\leqslant a_m\cos^m\theta$ should hold, by

$$\mathfrak{Re}\Bigg(\sum_{r=1}^{m}a_r(\beta+i\gamma)^k\Bigg)\leqslant\sum_{k=1}^{m}a_r\beta^k \iff |\theta(\beta+i\gamma)|\leqslant\theta(\beta),\quad\forall\,\beta\in\mathbb{R},\ \gamma\in\mathbb{R}.$$

But $a_4=\cdots=a_{n-1}=a_n=0$, by $a_m\cos(m\theta)-a_m\cos^m\theta\nleqslant 0$, $a_m\cos(m\theta)-a_m\cos^m\theta\ngeqslant 0$, if $m\geqslant 4$. And, $\log\phi(i\gamma)$, $\gamma\in\mathbb{R}$, is convex i.e. $-a_1\gamma+a_2\gamma^2-a_3\gamma^3$ is convex i.e. $a_3=0$, if $m=3$. Thus, $m\ngtr 2$. □

Theorem. Let

$$f(\eta)=\int_{\mathbb{R}^{nd}}e^{\eta^2y^2}\mu^{(t)}(dy) < \infty,\qquad\forall\,\eta>0,\ nd\geqslant 1$$

then characteristic ϕ is entire. And, if $\phi(k)\neq 0$, $\forall\,k\in\mathbb{C}$, then μ is Gaussian.

Proof. Follows by the theorem. ♡

Corollary. Let $Y^{(t)}_{\xi_\nu}+Y^{(t)}_{\eta_\omega}$ be Gaussian; then $Y^{(t)}_{\xi_\nu},Y^{(t)}_{\eta_\omega}$ are Gaussian iff i.i.d.

Proof. Follows by the prior theorem. ♡

Corollary (Kwapien, Pycia, Schachermayer; Bobkov, Houdre). Let $Y^{(t)}_M\in\mathbb{R}^{n\times n}$ be i.i.d random with the symmetric distribution

$$\mathbb{P}\left\{\left|\frac{1}{\sqrt{2}}\sum_{\xi=1}^{n}\sum_{\nu=1}^{d}Y^{(t)}_{\xi_\nu}\right|\geqslant x\right\}\leqslant\mathbb{P}\left\{\left|\sum_{\xi=1}^{n/2}\sum_{\nu=1}^{d}Y^{(t)}_{\sigma_{\xi_\nu}}\right|\geqslant x\right\},\quad\forall\,x\geqslant 0;\ 1_1=\sigma_{1_1}\neq\cdots\neq\sigma_{n_d}=n_d;$$

then $Y_{1_1},\ldots,Y_{n_d}$ are Gaussian random variables.

Proof. Assume $\mathbb{E}[{Y^{(t)}}^2]<\infty$ for all $(Y^{(t)}_{nd})$. Then, by symmetry and i.i.d,

$$\mathbb{E}\left[\left(\frac{1}{\sqrt{2}}\sum_{\xi=1}^{n}\sum_{\nu=1}^{d}Y^{(t)}_{\xi_\nu}\right)^2\right] = \mathbb{E}\left[\left(\sum_{\xi=1}^{n/2}\sum_{\nu=1}^{d}Y^{(t)}_{\xi_\nu}\right)^2\right].$$

In integration by part, inequality is equality; by symmetry $\frac{1}{\sqrt{2}}\sum_{\xi=1}^{n}\sum_{\nu=1}^{d}Y^{(t)}_{\xi_\nu}\overset{\mathrm{d}}{=}\sum_{\xi=1}^{n/2}\sum_{\nu=1}^{d}Y^{(t)}_{\xi_\nu}$.

Derivation. The prior is true for i.i.d $Y^{(t)}$ sequence $(Y^{(t)}_{\xi\eta}\,|\,\xi,\eta=1,\ldots,2^{n^2})$.

Definition. Let F_{n^2} be random measure. We say F_{n^2} converges (weakly) in probability resp. (strongly) almost surely for bounded continuous f if

$$\int f(dF_{n^2})\xrightarrow{\mathbb{P}}\int f(dF)\ \text{ resp. }\ \int f(dF_{n^2})\xrightarrow{\text{a.s.}}\int f(dF)\quad\text{as } n\longrightarrow\infty.$$

Theorem (equipartition asymptotics; entropy law). Differential entropy converges for i.i.d $Y^{(t)}_{1_1},\ldots,Y^{(t)}_{n_d}$ drawn by a density f; that is,

$$-\frac{1}{nd}\log f^{\otimes nd}(Y^{(t)}_{1_1},\ldots,Y^{(t)}_{n_d})\ \xrightarrow{\text{a.s.}}\ \mathbb{E}[-\log f(y)] = h(f) = -\int_{\mathbb{R}}f(y)\log f(y)\,dy.$$

Fig. 7: Entropy law (no cut-off) for affine density $f = 5 + \mathcal{N}(0,1)$

Proof. By strong law of large numbers with Riemann-Lebesgue convolution,

$$h(\mathcal{N}(\mu t, \sqrt{\Sigma} t)) \;=\; \frac{1}{2}\log\Big((2\pi t)^{nd}\det\Sigma\Big)\left(\frac{1}{2}\sum_{\xi=1}^{n}\sum_{\nu=1}^{d}\frac{\mathbb{E}[Y_{\xi_\nu}^{(t)^2}] - t^2\mu_{\xi_\nu}^2}{t\Sigma_{\xi_\nu\xi_\nu}}\right) \;=\; \frac{1}{2}\log\left(\det(2\pi t e\Sigma)\right).$$

$$\Longrightarrow \mathbb{P}\Big\{\mathbb{E}[-\log f] \;=\lim_{nd\longrightarrow\infty}\frac{1}{nd}\sum_{\xi=1}^{n}\sum_{\nu=1}^{d} -\log f(Y_{\xi_\nu}^{(t)})\int_{\mathbb{R}} f(y)\,dy \;=\; -\int_{\mathbb{R}} f(y)\log f(y)\,dy\Big\} = 1. \qquad \square$$

Remark. $d\mu_n \longrightarrow d\mu$ for all metrizable topology by Arzela-Ascoli theorem.

Theorem (maximum entropy). Let all density f be on support Ω. If

$$f^*(y) = f_\lambda = \exp\Big(\lambda_0 + \sum_{\gamma=1}^{m}\sum_{\xi_{\nu_1},\dots,\xi_{\nu_\gamma}}\lambda_{\xi_{\nu_1}\cdots\,\xi_{\nu_\gamma}}\bigotimes_{r=1}^{\gamma} y_{\xi_{\nu_r}}\Big) \;\Bigg|\; \begin{array}{l} y_{\xi_{\nu_r}}\otimes y_{\eta_{\omega_s}} = y_{\eta_{\omega_s}}\otimes y_{\xi_{\nu_r}},\ \forall\, r,s = 1,\dots,<\infty; \\ \forall\,\xi_{\nu_r},\eta_{\omega_s}\in\{\xi_\nu \mid \xi = 1,\dots,n;\ \nu = 1,\dots,d\}\end{array}$$

satisfy f constraints, then f^* uniquely maximizes entropy over all f.

Proof. Let $g \neq f^*$ satisfy f^* constraints; $\Omega_g \supseteq \Omega_{f^*}$ for finiteness, then

$$\begin{aligned} h(g) \;&=\; -\int_S g\log\Big(\frac{g}{f^*}\,f^*\Big)dy \;=\; \int_S g\log\Big(\frac{f^*}{g}\Big)dy \;-\int_S g\log f^*\,dy \\ &\leqslant\; -\int_S g\log f^*\,dy \;=\; -\int_S g\cdot\Big(\lambda_0 + \sum_{\gamma=1}^{m}\sum_{\xi_{\nu_1},\dots,\xi_{\nu_\gamma}}\lambda_{\xi_{\nu_1}\cdots\,\xi_{\nu_\gamma}}\bigotimes_{r=1}^{\gamma} y_{\xi_{\nu_r}}\Big)dy \\ &=\; -\int_S f^*\cdot\Big(\lambda_0 + \sum_{\gamma=1}^{m}\sum_{\xi_{\nu_1},\dots,\xi_{\nu_\gamma}}\lambda_{\xi_{\nu_1}\cdots\,\xi_{\nu_\gamma}}\bigotimes_{r=1}^{\gamma} y_{\xi_{\nu_r}}\Big)dy \;=\; -\int_S f^*\log f^*\,dy \;=\; h(f^*) \end{aligned}$$

where the inequality is an equality iff $g = f^*$ almost everywhere. $\square$

Derivation. Take $\Omega \equiv \mathbb{R}^{n\times n}$, $\lambda_{\xi_\nu}{=}\,\mathbb{E}[Y^{(t)}_{\xi_\nu}]/\mathbb{E}[Y^{(t)}_{\xi_\nu}, Y^{(t)}_{\xi_\nu}]$, $\lambda_{\xi_\nu\eta_\omega}{=}-1/(2\,\mathbb{E}[Y^{(t)}_{\xi_\nu}, Y^{(t)}_{\eta_\omega}])$, then

$$f^*(y) = \exp\Big(\lambda_0 + \sum_{\xi=1}^{n}\sum_{\nu=1}^{d}\lambda_{\xi_\nu}\, y_{\xi_\nu} + \sum_{\xi=1}^{n}\sum_{\nu=1}^{d}\sum_{\eta=1}^{n}\sum_{\omega=1}^{d}\lambda_{\xi_\nu\eta_\omega}\, y_{\xi_\nu}{\otimes} y_{\eta_\omega}\Big),\ \text{resp.}\ \exp\big(\lambda_0 + \lambda y + \widetilde{\lambda} y^2\big)\ \text{for}\ n{=}d{=}1.$$

Remark. Gaussians are maximum entropy distribution for tree structures.

Theorem (complexity). For $0\leqslant\varepsilon<1$, i.i.d $Y_M=(Y_{1_1},\ldots,Y_{1_d},\ldots,Y_{n_1},\ldots,Y_{n_d})\sim\mathcal{N}(t\mu,\sqrt{t\Sigma})$,

$$\mathbb{P}\big[Y>(1+(\mathrm{sgn})\varepsilon)\,\mathbb{E}[Y]\big] \leqslant \exp\Big(-nd\,\frac{\varepsilon^2}{\gamma}\,\mathbb{E}[Y]\Big) = \begin{cases} e^{-\mathcal{O}(1)} \Longrightarrow \varepsilon^2 = \mathcal{O}(1/ndt\mu) \\ \frac{1}{(\mathbb{E}[Y])^{-\frac{1}{c}}} \Longrightarrow \varepsilon = \mathcal{O}\Big(\sqrt{(1/ndt\mu)\log(1/ndt\mu)}\Big) \end{cases}\Bigg|\ q = \begin{cases} 3\ \text{if}\ (\mathrm{sgn}) = +1 \\ 2\ \text{if}\ (\mathrm{sgn}) = -1. \end{cases}$$

Proof. Similar[3] to $\frac{1}{nd}S^d_n\sim\mathcal{N}(t\mu,\sqrt{t\Sigma}/\sqrt{nd})$ and $S^d_n\sim\mathcal{N}(ndt\mu,\sqrt{ndt\Sigma})$, in the Markov's inequality for continuous non-decreasing $f(Y)=Y^2$ Chebyshev:

$$\frac{\mathbb{E}[Y]}{\lambda} > \mathbb{P}[Y>\lambda] = \mathbb{P}[f(Y)>f(\lambda)] < \frac{\mathbb{E}[f(Y)]}{f(\lambda)} \Longrightarrow \mathbb{P}[|Y-\mathbb{E}[Y]|\geqslant\lambda] = \mathbb{P}[(Y-\mathbb{E}[Y])^2\geqslant\lambda^2] \leqslant \frac{\mathbb{E}[(Y-\mathbb{E}[Y])^2]}{\lambda^2} = \frac{\mathrm{Var}[Y]}{\lambda^2}$$

the probability of $Y^{(t)}$ deviating from $\mathbb{E}[Y^{(t)}]$ tends Chernoff $f(Y^{(t)})=e^{xY^{(t)}}$:

$$\mathbb{P}[Y^{(t)}>(1{+}\varepsilon)\mathbb{E}[Y^{(t)}]] = \mathbb{P}[e^{xY^{(t)}} > e^{(1+\varepsilon)\,x\,\mathbb{E}[Y^{(t)}]}] \leqslant \frac{\mathbb{E}[\exp(x\sum_{\xi=1}^{n}\sum_{\nu=1}^{d}Y^{(t)}_{\xi_\nu})]}{\exp((1{+}\varepsilon)\,x\sum_{\xi=1}^{n}\sum_{\nu=1}^{d}\mathbb{E}[Y^{(t)}_{\xi_\nu}])} = \frac{\exp(ndt\mu x + \frac{1}{2}ndt\Sigma x{\otimes}x)}{\exp((1{+}\varepsilon)ndt\mu x)}$$

i.e. $\mathbb{P}[Y^{(t)}>(1{+}\varepsilon)\mathbb{E}[Y^{(t)}]] \leqslant \exp\Big(nd\Big(\frac{1}{2}t\Sigma x{\otimes}x + t\mu x - (1{+}\varepsilon)t\mu x\Big)\Big)$.

Now, to obtain the tight bound, for non-singular Σ, set

$$t\Sigma x{\otimes}x \longrightarrow 0,\quad t=\ln(1{+}\varepsilon)\qquad \forall\,(\varepsilon{\neq}0).$$

Then, about $\varepsilon_0{=}0$,

$$(1{+}\varepsilon)\ln(1{+}\varepsilon)\ \geqslant\ \varepsilon+\frac{\varepsilon^2}{2}-\frac{\varepsilon^3}{6}\ \geqslant\ \varepsilon+\frac{\varepsilon^2}{3};\qquad \varepsilon\ \geqslant\ \varepsilon-\frac{\varepsilon^2}{2}+\frac{\varepsilon^2}{6}\ \geqslant\ \varepsilon-\frac{\varepsilon^2}{2}+\frac{\varepsilon^3}{6};\qquad \forall\ 0\leqslant\varepsilon<1$$

where

$$(1{+}\varepsilon)\Big(\sum_{\xi\geqslant1}(-1)^{\xi-1}\frac{(\varepsilon-\varepsilon_0)^\xi}{(1+\varepsilon_0)^\xi}\cdot\frac{(\xi-1)!}{\xi!}\Big)\ =\ \varepsilon+\sum_{\xi\geqslant2}(-1)^\xi\varepsilon^\xi\Big(-\frac{1}{\xi}+\frac{1}{\xi-1}\Big).$$

Similarly, setting $t=\ln(1{-}\varepsilon)$, about $\varepsilon_0{=}0$,

$$(1{-}\varepsilon)\ln(1{-}\varepsilon)\ \geqslant\ -\varepsilon+\frac{\varepsilon^2}{2}+\frac{\varepsilon^3}{6}\ \geqslant\ -\varepsilon+\frac{\varepsilon^2}{2};\qquad -\varepsilon\ \geqslant\ -\varepsilon-\frac{\varepsilon^2}{2}\ \geqslant\ -\varepsilon-\frac{\varepsilon^2}{2}-\frac{\varepsilon^3}{6};\qquad \forall\ 0\leqslant\varepsilon<1$$

where

$$(1{-}\varepsilon)\Big(\sum_{\xi\geqslant1}-\frac{(\varepsilon-\varepsilon_0)^\xi}{(1-\varepsilon_0)^\xi}\cdot\frac{(\xi-1)!}{\xi!}\Big)\ =\ -\varepsilon+\sum_{\xi\geqslant2}\varepsilon^\xi\Big(-\frac{1}{\xi}+\frac{1}{\xi-1}\Big).$$

Therefore, the conclusion follows as required. □

Derivation (discrete). Let X be n-trial, m-type $\mathbb{R}$-valued, then by $\mu_\xi=p_\xi$, $\Sigma_{\xi\eta}=\mathrm{Cov}(X_\xi,X_\eta)$, not necessarily independent, $\phi(x)$ equals

$$\sum_{\sum_{\xi=1}^{m}n_\xi=n}\frac{n!}{\prod_{\xi=1}^{m}n_\xi!}\bigotimes_{\xi=1}^{m}\Big(\sum_{\gamma=0}^{\infty}\frac{1}{\gamma!}\,\frac{d^\gamma}{dz_\xi^\gamma}\,e^{iy_\xi x_\xi}\big|_{x_\xi=x_0,\,y_\xi=1}\,x_\xi^\gamma\ p(y_\xi{=}1)\Big)^{n_\xi} = \Big(\sum_{\xi=1}^{m}e^{ix_\xi}\,p_\xi\Big)^n$$

$$\text{For } m{=}2:\quad \phi(x) = \sum_{\gamma=0}^{n}\frac{n!}{\gamma!(n{-}q)!}\big(\exp(ix)\,p\big)^\gamma(1{-}p)^{n-q}\ =\ \big(\exp(ix)\,p+1-p\big)^n.$$

[3] given $\mathbb{E}[e^{(\langle x,y\rangle)}]=\big(\exp(t\mu x+\frac{1}{2}t\Sigma x{\otimes}x)\big)^{nd}=\exp(ndt\mu x+\frac{1}{2}ndt\Sigma x{\otimes}x)$, the moment generating function of the normal random variable with mean $ndt\mu$ and variance $ndt\Sigma$.

Theorem (bivariate). Let $(Y_{nd}) = (Y_{\xi_\nu} \in \{0, \varepsilon\}) \mid p_{\xi_\nu} = p(Y_{\xi_\nu} = c) = \mathbb{E}[Y_{\xi_\nu}]$.

$$\lim_{c \longrightarrow 1} \mathbb{P}\big[Y > (1 + (\text{sgn})\varepsilon)\, \mathbb{E}[Y]\big] \;\leqslant\; \exp\Big(-\,\frac{\varepsilon^2}{\gamma}\, \mathbb{E}[Y]\Big) \;\Big|\; q = \begin{cases} 3 & \text{if } (\text{sgn}) = +1 \\ 2 & \text{if } (\text{sgn}) = -1. \end{cases}$$

Proof. By $(1+Y) \leqslant \exp(Y)$, for t, and independence for 4th equality below,

$$\mathbb{P}[Y > (1+\varepsilon)\mathbb{E}[Y]] = \mathbb{P}[e^{tY} > e^{(1+\varepsilon)\, t\, \mathbb{E}[Y]}] \leqslant \frac{\mathbb{E}[\exp(tY)]}{\exp((1+\varepsilon)\, t\, \mathbb{E}[Y])} = \frac{\mathbb{E}\big[\exp\big(t \sum_{\xi=1}^{n} \sum_{\nu=1}^{d} Y_{\xi_\nu}\big)\big]}{\exp\big((1+\varepsilon)\, t \sum_{\xi=1}^{n} \sum_{\nu=1}^{d} \mathbb{E}[Y_{\xi_\nu}]\big)}$$

$$= \frac{\mathbb{E}\big[\prod_{\xi=1}^{n} \prod_{\nu=1}^{d} \exp(tY_{\xi_\nu})\big]}{\exp\big((1+\varepsilon)\, t\, \mathbb{E}[Y]\big)} = \frac{\prod_{\xi=1}^{n} \prod_{\nu=1}^{d} \mathbb{E}[\exp(tY_{\xi_\nu})]}{\exp\big((1+\varepsilon)\, t\, \mathbb{E}[Y]\big)} \leqslant \frac{\prod_{\xi=1}^{n} \prod_{\nu=1}^{d} \big(p(Y_{\xi_\nu} = 0) \cdot e^{(t \,\cdot\, (Y_{\xi_\nu} = 0))} + p(Y_{\xi_\nu} = c) \cdot e^{(t \,\cdot\, (Y_{\xi_\nu} = c))}\big)}{\exp\big((1+\varepsilon)\, t\, \mathbb{E}[Y]\big)}$$

$$= \frac{\prod_{\xi=1}^{n} \prod_{\nu=1}^{d} \big(1 + (e^{tc} - 1)\, p_{\xi_\nu}\big)}{\exp\big((1+\varepsilon)\, t\, \mathbb{E}[Y]\big)} \leqslant \frac{\prod_{\xi=1}^{n} \prod_{\nu=1}^{d} \exp\big((e^{tc} - 1)\, p_{\xi_\nu}\big)}{\exp\big((1+\varepsilon)\, t\, \mathbb{E}[Y]\big)} = \frac{\exp\big(\sum_{\xi=1}^{n} \sum_{\nu=1}^{d} (e^{tc} - 1)\, p_{\xi_\nu}\big)}{\exp\big((1+\varepsilon)\, t\, \mathbb{E}[Y]\big)} = \frac{\exp\big((e^{tc} - 1)\, \mathbb{E}[Y]\big)}{\exp\big((1+\varepsilon)\, t\, \mathbb{E}[Y]\big)}.$$

That is,

$$\mathbb{P}\big[Y > (1+\varepsilon)\, \mathbb{E}[Y]\big] \;\leqslant\; \exp\big((e^{tc} - 1 - (1+\varepsilon)\, t)\, \mathbb{E}[Y]\big)\,.$$

Then, by $t = \ln(1+\varepsilon)$, and expansion of $(1+\varepsilon)\ln(1+\varepsilon)$ about $\varepsilon_0 = 0$,

$$\mathbb{P}\big[Y > (1+\varepsilon)\, \mathbb{E}[Y]\big] \leqslant \exp\Big(\Big(\,(1+\varepsilon)^c - 1 - \varepsilon - \frac{\varepsilon^2}{3}\Big)\, \mathbb{E}[Y]\Big), \quad \forall\, (\varepsilon \geqslant 0).$$

Respectively, by $t = \ln(1-\varepsilon)$, and expansion of $(1-\varepsilon)\ln(1-\varepsilon)$ about $\varepsilon_0 = 0$,

$$\mathbb{P}\big[Y > (1-\varepsilon)\, \mathbb{E}[Y]\big] \leqslant \exp\Big(\Big(\,(1-\varepsilon)^c - 1 + \varepsilon - \frac{\varepsilon^2}{2}\Big)\, \mathbb{E}[X]\Big), \quad \forall\, (\varepsilon \geqslant 0).$$

as required. □

Definition. Density f is Kolmogorov-Smirnov if:

$$\lim_{N \longrightarrow \infty} f\Bigg(0 \;<\; \Bigg(\sup_{-\infty < x < \infty} \Big|\, F_N[x] \;-\; F_N^*[x] \,\Big|\Bigg) \sqrt{\frac{N}{2}} \;<\; z\Bigg) \;=\; K(z) \tag{8}$$

$$K(z) = \sum_{\gamma = -\infty}^{\infty} (-1)^\gamma \exp(-2\gamma^2 z^2). \tag{9}$$

Theorem (empirical). Let $\mathbb{E}[\xi_N] \longrightarrow \infty$ and $(\mathbb{E}[\xi_N])^{-1}\, \mathrm{Var}[\xi_N] < \infty$, for

$$\begin{aligned} F_N^*[x] \;&=\; (\mathbb{E}[\xi_N])^{-1} \sum_{\eta=1}^{N} \mathbb{1}_{\{-\infty,\, x\}}(Y_{\xi_\eta}^{(t)}) \\ &\equiv\; 0 \;\text{ if }\; \xi_N = 0 \end{aligned} \quad \left|\; \begin{aligned} &\mathbb{1}_{\{-\infty,\, x\}}(Y_{\xi_\eta}^{(t)}) = \begin{cases} 1 & \text{if } Y_{\xi_\eta}^{(t)} \leqslant y \\ 0 & \text{otherwise} \end{cases} \\ &X \sim F(Y^{(t)}),\, \forall\, x \in \mathbb{R} \end{aligned} \right. \tag{10}$$

where label ξ_N is independent of $Y_{\xi_\eta}^{(t)} \,|\, \eta = 1, \ldots, N$. Then[4], as $\varepsilon \longrightarrow 0$,

$$\lim_{N \longrightarrow \infty} (\mathbb{E}[\xi_N])^{-1} N \;\lim_{\varepsilon \longrightarrow 0}\; \sup_{0 \leqslant x \leqslant \frac{1}{2}} \ln(g(x, \varepsilon)) \;\overset{d}{\sim}\; \lim_{N \longrightarrow \infty} -\,\frac{\varepsilon^2}{2\,\big(1 + (\mathbb{E}[\xi_N])^{-1}\, \mathrm{Var}[\xi_N]\big)}. \tag{11}$$

Proof. For $\varepsilon \ll$, $\varepsilon > 0$, $N > 1$, let:

$$K_N^- = \sup_{x \geqslant 0} \big(1 - \mathbb{P}_N^*[x] - \mathbb{P}_N^*[-x]\big), \quad K_N^+ = \sup_{x \geqslant 0} \big(\mathbb{P}_N^*[x] + \mathbb{P}_N^*[-x] - 1\big) \tag{12}$$

$$K_N = \sup_{x \geqslant 0} \big(\big|\, 1 \;-\; \mathbb{P}_N^*[x] \;-\; \mathbb{P}_N^*[-x] \,\big|\big) = \max\big(K_N^+, K_N^-\big) \tag{13}$$

$$\text{i.e. } f(K_N^+ \geqslant \varepsilon) \leqslant f(K_N \geqslant \varepsilon) \leqslant \big(f(K_N^+ \geqslant \varepsilon) + f(K_N^- \geqslant \varepsilon)\big). \tag{14}$$

[4] where $p = \sup(S \subseteq T) \iff p = \mathrm{lub}\,(S \subseteq T) := p \in T \mid p - \varepsilon < x \leqslant p,\, \forall\, x \in S,\, \varepsilon > 0$; moreover, $p = \inf(S \subseteq T) \iff p = \mathrm{glb}\,(S \subseteq T) := p \in T \mid p \leqslant x < p + \varepsilon,\, \forall\, x \in S,\, \varepsilon > 0$.

WLOG, $\xi_{N>1}$ equals sum of i.i.d (i.e. same probability space): $\mathbb{E}[\xi_N] = N\mathbb{E}[\xi]$, $\xi_N = N\xi$, with generating functions φ_N, $\varphi \mid \varphi_N(\mathcal{Z}) = \mathbb{E}[\mathcal{Z}^{\xi_N}] = \varphi^N(\mathcal{Z})$ implies

$$\lim_{N\longrightarrow\infty} (N\,\mathbb{E}[\xi])^{-1}\ \ln(f(K_N^+ \geqslant \varepsilon)) \ = \ (\mathbb{E}[\xi])^{-1} \sup_{0\leqslant x\leqslant \frac{1}{2}} \ln(g(x,\varepsilon)). \tag{15}$$

Thus,

$$\lim_{\varepsilon\longrightarrow 0}\ (\mathbb{E}[\xi])^{-1} \sup_{0\leqslant x\leqslant \frac{1}{2}} \ln g(x,\varepsilon) \ \sim \ -\ \frac{\varepsilon^2}{2\Big(1+\lim\limits_{N\longrightarrow\infty} (\mathbb{E}[\xi_N])^{-1}\,\mathrm{Var}[\xi_N]\Big)} \tag{16}$$

as required. □

In general, $X_{\xi_\eta} \mid 1 \leqslant \eta \leqslant N$ uniformly distributed on $[0,1]$ implies

$$K_N^+ \ = \sup_{0\leqslant x\leqslant 1/2} \Big(\mathbb{P}_N^*[x] + \mathbb{P}_N^*[1-x] - 1\Big), \qquad \forall\, x \in [0,\ \frac{1}{2}]. \tag{17}$$

That is,

$$f(K_N^+ \geqslant \varepsilon)\ \geqslant\ f\Big(\mathbb{P}_N^*[x] + \mathbb{P}_N^*[1-x] - 1\ \geqslant\ \varepsilon\Big) =$$

$$= f\Big(N^{-1} \sum_{\eta=1}^{\xi_N} \Big(\mathbb{1}_{\{-\infty,\,x\}}(X_{\xi_\eta}) + \mathbb{1}_{\{-\infty,\,1-x\}}(X_{\xi_{\xi_\eta}})\Big)\ \geqslant\ (1+\varepsilon)\ \mathbb{E}[\xi]\Big). \tag{18}$$

Using the $\Xi_{\xi_N} = \sum_{\eta=1}^{N} \big(\mathbb{1}_{\{-\infty,\,x\}}(X_{\xi_\eta}) + \mathbb{1}_{\{-\infty,\,1-x\}}(X_{\xi_\eta})\big)$ moment generating function

$$\mathbb{E}[\exp(\mathcal{Z}^{\xi_N})] = \mathbb{E}\Big[x\exp(2\mathcal{Z}) + (1-2x)\exp(\mathcal{Z}) + x\Big] = \varphi^N\Big(x\exp(2\mathcal{Z}) + (1-2x)\exp(\mathcal{Z}) + x\Big) \tag{19}$$

then, if Y_{ξ_1} and Y_{ξ_2} are independent with m.g.f $\Psi(\mathcal{Z}) = \varphi\big(x\exp(2\mathcal{Z})\ +\ (1-2x)\exp(\mathcal{Z})\ +\ x\big)$,

$$f\Big(N^{-1}\,\Xi_{\xi_N}\ \geqslant\ (1+\varepsilon)\,\mathbb{E}[\xi]\Big)\ \ =\ \ f\Big(N^{-1}\sum_{\eta=1}^{N} Y_{\xi_\eta}\ \geqslant\ (1+\varepsilon)\,\mathbb{E}[\xi]\Big). \tag{20}$$

By Chernoff's method of Markov's inequality,

$$\ln g(x,\varepsilon) = \lim_{N\to\infty} N^{-1} \ln f\Big(N^{-1}\sum_{\eta=1}^{N} Y_{\xi_\eta} \geqslant (1+\varepsilon)\,\mathbb{E}[\xi]\Big) \tag{21}$$

$$g(x,\varepsilon) = \ \Psi(\tau)\ \exp\Big(-\,\tau\,(1+\varepsilon)\,\mathbb{E}[\xi]\Big)\ \ \Big|\ \Psi'(\tau) - (1+\varepsilon)\,\Psi(\tau)\,\mathbb{E}[\xi] = 0. \tag{22}$$

That is,

$$\lim_{N\longrightarrow\infty} \frac{1}{N\,\mathbb{E}[\xi]}\ \ln(f(K_N^+ \geqslant \varepsilon))\ =\ \frac{1}{\mathbb{E}[\xi]}\ \sup_x\ \ln(g(x,\varepsilon)) \tag{23}$$

which is an equality with the unique continuous solution $\tau = \tau(x,\varepsilon)$, for $\varepsilon \ll$, by implicit function theorems; in particular, in elementary derivations, for $\varepsilon \longrightarrow 0$,

$$\tau\ =\ \frac{\varepsilon}{2x+\beta}\ +\ \mathcal{O}(\varepsilon). \tag{24}$$

And, substituting, as $\varepsilon \longrightarrow 0$,

$$\ln(g(x,\varepsilon))\ =\ -\,\frac{\varepsilon^2\,\mathbb{E}[\xi]}{2x+\beta}\ +\ \frac{(2x\,\mathbb{E}[\xi] + \mathrm{Var}[\xi])\ \varepsilon^2}{2\,(2x+\beta)^2}\ +\ \mathcal{O}(\varepsilon^2). \tag{25}$$

By continuity of $\ln g(x,\varepsilon)$, based on the sum of variables,

$$\sup_x\ \lim_{\varepsilon\longrightarrow 0}\ (\varepsilon^2\,\mathbb{E}[\xi])^{-1}\ \ln(g(x,\varepsilon))\ = \tag{26}$$

$$=\ \sup_x\Big(-\ \frac{1}{2\,(2x+\beta)}\Big)\ \ =\ \ \frac{1}{2\,(2x+\beta)}. \tag{27}$$

Enveloping $\mathcal{U}(\mathfrak{gl}_n^{\otimes})/\mathcal{O}(n)$ wild forest

Self consistent stationary drifting

From [6], let random two-sided Kolmogorov-Smirnov statistic[5], for random distance ρ between continuous distributions ($\mathbb{P}_N$, $\mathbb{P}_N^*$), of N samples, on electroencephalogram (EEG), be:

$$D_N \;=\; \sup_{-\infty < x < \infty} \big|\, \mathbb{P}_N[x] \;-\; \mathbb{P}_N^*[x] \,\big|.$$

Then, in symmetry hypothesis $\mathcal{H}_0 : 1 - F(y) - F(-y) = 0$,

$$\lim_{N \longrightarrow \infty} f\Big(0 < D_N < \varepsilon\Big) \quad = \quad K\left(\sqrt{\frac{N}{2}}\,\varepsilon\right) \tag{28}$$

where RHS is the Kolmogorov function

$$K(z) \;=\; \sum_{\gamma=-\infty}^{\infty} (-1)^{\gamma} \exp(-2\gamma^2 z^2) \;=\; \lim_{N \longrightarrow \infty} f\left(0 < \sqrt{\frac{N}{2}}\, D_N < z\right). \tag{29}$$

Thus, the significance level of D_N equals

$$1 - K\left(\sqrt{\frac{N}{2}}\,\varepsilon\right) \tag{30}$$

where significance level and confidence interval of $\sqrt{\frac{N}{2}}\, D_N$ are respectively:

$$1 - K(\varepsilon) \tag{31}$$

$$(0,\, \varepsilon). \tag{32}$$

By continuous, monotonically increasing Kolmogorov function $K(z)$ and sample cumulative distribution (SCD) $G(\rho)$ on $[0, 1]$, for all ρ, the equation

$$\varepsilon \;=\; 1 - K\left(\sqrt{\frac{N}{2}}\,\varepsilon\right) \tag{33}$$

has a unique solution ε^* for all ρ.

Moreover, solution ε^*, (33), is stationary-SCD independent, differs upwards in distribution under nonstationary SCD; and, the criterion for accepting symmetry hypothesis $\mathcal{H}_0 : 1 - F(y) - F(-y) = 0$ is the max between significance level and confidence interval for set of permissible outcomes [5]; i.e. solution ε^*, (33), corresponds to significance level or confidence interval.

Remark. A time series Y_t is stationary over an interval if no change occurs, that is,

$$E[Y^{(t)}] = E[Y^{(t+\tau)}]\,, \qquad \sigma^2[Y^{(t)}] = \sigma^2[Y^{(t+\tau)}]. \tag{34}$$

Let $G_N(\rho)$ be empirical CDF for distances $\rho(N)$ of two non-crossing length N samples, given by

$$\rho(N) = \|F_{1,N}(x) - F_{2,N}(x)\|_C\,. \tag{35}$$

In nonstationary SCD (from nonstationary, N samples), then

$$G_N(\rho) = 1 - \rho \tag{36}$$

gives numerical solution $\rho^*(N)$ of self consistent stationary drifting SCSD i.e. probability $\rho > \rho^*$. In particular, for a detector (SCSD) by X_t.

[5] In practice, the Kolmogorov-Smirnov's statistic is applicable for $N \geqslant 50$ [see 4, p.215].

$2^{\aleph_c}$ quarternion algebra for supersolvable extremal theory

The Kolmogorov superalgebra of $\mathbb{R}\cup\{\infty\}$ non-decreasing maximum-entropy $(\phi_{\xi|\eta})_{\eta\geqslant h(x)}$ process, of independent-nonstationary increments on $\mathbb{P}_x$ Lifshitz continuity, implies strong Markov property.

In particular, fixing a quantum algorithm or walk as event sequence, where output at every time is estimate on supersymmetry of events thus far. That is, the algorithm tree maintains single integer n at each node supporting, up to affine transformations, the two operations:

(1). update(): increments n by 1

(2). query(): must output (an estimate of) n.

Let n start at 0, for nonsingular transformations. Indeed, trivial algorithm maintains n using $\lceil \log n \rceil$ bits of memory (a counter); the goal here is to use much less space. It is not hard to prove it is impossible to use exact $O(\log n)$ bits of space. That is, query() is answered with an n estimate $\bar{n}$ which satisfy condition $\mathbb{P}[|\bar{n}-n|>\varepsilon n]<c$ $(0<\varepsilon, c<1)$ given to the algorithm up front. Likewise, the algorithm gives an estimate for ε, c as shown below. The algorithm works as follows: first, initialize $Y \longleftarrow 0$. Then, for each update, "increment" Y by probability $\frac{1}{2^Y}$; for each query, output $\bar{n}=2^Y-1$. Intuitively, variable Y is attempting to store a value that is $\approx log 2n$.

Moreover, to decrease failure probability of the algorithm, we instantiate s independent copies of classical algorithm and average their outputs. Thus, we obtain independent estimates $\bar{n}_1, \ldots, \bar{n}_m$ from independent instantions of the algorithm. The output to a query is then given by

$$\bar{n} \ = \ \frac{1}{m}\sum_{i=1}^{m}\bar{n}_i \tag{37}$$

As each $\bar{n}_i$ is unbiased estimate of n, so is the average $\bar{n}$. By independent random variables, and by ρ^2 multiple of variance from constant $\rho = 1/m$ multiple of a random variable, then

$$P(|\bar{n}-n| > \varepsilon n) \ < \ \frac{1}{2m\varepsilon^2} \ < \ c \tag{38}$$

for all $m > 1/(2\varepsilon^2 c) = \Theta(1/(\varepsilon^2 c))$.

In particular, in quantum algorithm case, simple method to reduce dependence on failure probability c from $1/c$ to $log(1/c)$ follows: we run t instantions of the classical algorithm, each with failure probability $\frac{1}{3}$, i.e. for each, $m = \Theta(1/\varepsilon^2)$. We then output median estimate from the m classical instantions. Note that the expected number of classical instantions that succeed is at least $2t/3$. For the median to be a bad estimate, at most half the classical instantions can succeed, i.e. the number of succeeding instantiations deviated from the expectation by at least $t/6$. Define

$$Y_i = \begin{cases} 1 & \text{if the ith classical instantiation succeeds} \\ 0 & \text{otherwise.} \end{cases} \tag{39}$$

Then by the Chernoff bound,

$$\mathbb{P}\Big(\sum_i Y_i \leqslant \frac{t}{2}\Big) \leqslant \mathbb{P}\Big(\Big|\sum_i Y_i - \mathbb{E}\sum_i Y_i\Big| \geqslant \frac{t}{6}\Big) \leqslant 2e^{-t/3} < c \tag{40}$$

for $t \in \Theta(\log(1/c))$.

In terms of overall space complexity, the quantum algorithm shows that it is running a total of $mt=\Theta(\log(1/c)/\varepsilon^2)$ instantions of classical algorithm.

Now note that once an algorithm counter Y reaches value $\log(mtn/c)$, the probability that it is incremented at any given time is at most $c/(mtn)$. Thus, the probability that it is incremented in next n increments is at most $c/(mt)$; i.e. by union bound, with probability $1 - c$, none of the mt classical instantions ever stores value larger than $\log(mtn/c)$ which takes $O(\log\log(mtn/c))$ bits. Thus, total space complexity is, with probability $1-c$, at most

$$O(\varepsilon^{-2}\log(1/c)(\log\log(n/(\varepsilon c)))). \tag{41}$$

In particular, for constant ε, c (say each 1/100), the total space complexity is $O(\log\log n)$ with constant probability. This is exponentially better than the $\log n$ space achieved by storing a counter!

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